Enumerative geometry of double spin curves

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Give a small boy a hammer, and he will find that everything he encounters needs pounding.

The Conduct of Inquiry
Abraham Kaplan

This one is for my parents who made me curious,
and for my wife who made me happy.

Abstract

This thesis has two parts. In Part I we consider the moduli spaces of curves with multiple spin structures and provide a compactification using geometrically meaningful limiting objects. We later give a complete classification of the irreducible components of these spaces. The moduli spaces built in this part provide the basis for the degeneration techniques required in the second part.

In the second part we consider a series of problems inspired by projective geometry. Given two hyperplanes tangential to a canonical curve at every point of intersection, we ask if there can be a common point of tangency. We show that such a common point can appear only in codimension 1 in moduli and proceed to compute the class of this divisor. We then study the general properties of curves in this divisor.

Our divisor class has small enough slope to imply that the canonical class of the moduli space of curves with two odd spin structures is big when the genus is greater than 9. If the corresponding coarse moduli spaces have mild enough singularities, then they have maximal Kodaira dimension in this range.

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Contents

Conventions

$\mathcal{M}_g$	moduli stack of smooth curves of genus g
$\overline{\mathcal{M}}_g$	moduli stack of stable spin curves of genus g
$\overline{M}_g$	coarse moduli scheme of $\overline{\mathcal{M}}_g$
$\mathcal{S}_g^-$	the moduli stack of odd spin curves of genus g
$\mathcal{S}_g^+$	the moduli stack of even spin curves of genus g
$\mathcal{S}_g^{--}$	moduli stack of double odd spin curves
$\overline{\mathcal{S}}_g^-$	and also $\overline{\mathcal{S}}_g^+$, $\overline{\mathcal{S}}_g^{--}$ are the respective compactifications
$\overline{S}_g^-$	coarse moduli scheme of $\overline{\mathcal{S}}_g^-$
$\overline{S}_g^{--}$	coarse moduli scheme of $\overline{\mathcal{S}}_g^{--}$
$\lvert V\rvert$	for a vector space V, is the projective space of lines on V
$\mathbb{P}(V)$	for a vector space V, is the projective space of 1-dimensional quotients of V
$\lvert\mathcal{L}\rvert$	for a line bundle $\mathcal{L}$ stands for $\lvert\mathrm{H}^0(\mathcal{L})\rvert$
$\mathbb{P}(\mathcal{E})$	for a rank-1 sheaf $\mathcal{E}$ only, will denote $\underline{\mathrm{Proj}}(\mathrm{Sym}^* \mathcal{E})$
S_C^-	the set of odd theta characteristics on C
S_C^+	the set of even theta characteristics on C
$\sqrt{L}$	the set of line bundles with tensor square L
$[n]$	the set of integers $\{1,\dots,n\}$
$[m,n]$	the set of integers $\{m,\dots,n\}$
$\#U$	for a finite set U, is the number of elements in U
S_C^-	for a curve C is the set of odd theta characteristic of C
S_C^+	for a curve C is the set of even theta characteristic of C

Chapter 1

Preface

As opposed to the main body of work, in this preface we wish explain the motivation behind the main results as informally as possible. In the meantime, we will introduce our point of view of the subject and present the main results of the thesis. The first section is written with an undergraduate in mind as a potential reader and then we gradually pick up pace.

1 Curves and their deformations

Algebraic curves lie at the heart of algebraic geometry, not least because they are simple enough to be studied in depth, but also because they are rich enough to interact with complicated objects non-trivially, shedding some light into their complexity. The topic of algebraic curves is remarkable in yet another way: it is one of the oldest subjects in mathematics.

Initially, an algebraic curve was the zero set of a two variable polynomial $f \in \mathbb{R}[x, y]$. Over time, it became clear that algebra works much better over $\mathbb{C}$ than over $\mathbb{R}$ and geometry works much better if the spaces are compact. Therefore, by 19th century, a curve was the zero locus of a homogeneous polynomial $f \in \mathbb{C}[x, y, z]$ in the projective plane $\mathbb{P}^2_{\mathbb{C}}$.

In working with such $f \in \mathbb{C}[x, y, z]$, it is natural to vary the coefficients of the polynomial by defining a function $F : \mathbb{C} \to \mathbb{C}[x, y, z]$ with $F(0) = f$. Then the zero locus of $F(t)$ is a plane curve $C_t \subset \mathbb{P}^2_{\mathbb{C}}$ and we witness a deformation of curves as t varies.

We can cut out curves in a higher dimensional projective space $\mathbb{P}^n_{\mathbb{C}}$ by using more polynomials: $f_1, \ldots, f_m \in \mathbb{C}[x_0, \ldots, x_n]$. If we take $m = n - 1$ then in general the intersection locus will indeed be 1 dimensional and smooth. However, very few curves appear in this way. If $f_1, \ldots, f_{n-1}$ are taken so that their common locus of intersection consists of the union of many curves, then we would need to use more polynomials to single out the component we need. It turns out that most curves can only be obtained in this way by using m polynomials with $m \geq n$.

On the other hand, given m equations taken at random, then the intersection locus will be zero dimensional if $m = n$ and empty if $m > n$. Therefore, if we start with m

equations cutting out a curve, any general perturbation of the coefficients would cease to give a curve. However, a far stronger statement is true. For most curves, we will not only require $m \geq n$ equations but also *any* change in the coefficients that are polynomial with respect to a free parameter $t \in \mathbb{C}$ will destroy the curve. Thus, the ease with which we deformed plane curves was in fact misleading about the difficulties lying ahead.

One way to avoid these difficulties, is to unshackle a curve from its defining equations. Curves became stand-alone objects freed from an ambient space through the work of Riemann. In modern terminology, we can define an abstract (and smooth) curve as follows:

Definition 1. An abstract curve is a compact complex manifold of complex dimension 1.

It is clear that a smooth algebraic curve cut out by polynomials will yield an abstract curve, however it is a highly non-trivial fact due to Riemann that an abstract curve can in fact be realized as the zero set of polynomials in projective space. Having had difficulties deforming embedded curves, we can now deform curves in the abstract by considering a well behaved morphisms between two complex manifolds, $\pi : \mathcal{X} \to B$, such that each fiber is a curve.

Underlying each curve C is a real oriented compact real surface S. Such a surface can topologically be completely classified by a non-negative integer g, the genus of S and of C. A sphere has genus 0, a torus has genus 1 and we say S has genus g if we need to glue g copies of a torus before it resembles S.

A more algebraic way to define the genus is to realize that given a curve C the space of global holomorphic differentials on C, denoted $\mathrm{H}^0(C, \omega_C)$, has finite complex dimension. Non-trivially, the genus equals the dimension of this vector space, that is, $g = \dim_{\mathbb{C}} \mathrm{H}^0(C, \omega_C)$.

The basic intuition after some experimentation is that the genus of a curve should remain invariant under deformations. In practice, this becomes essentially built in to the definition of a deformation. However, it was not at all clear whether any two curves of the same genus could be deformed to one another and this problem was of crucial importance.

Let us say that two curves C_1 and C_2 are deformation equivalent if one can be deformed to the other. One can often prove results to the effect that C_1 and C_2 share a given geometric property if they are deformation equivalent. Moreover, we can often construct simple curves where these geometric properties can be checked explicitly. If we knew that any two curves of the same genus are equivalent under deformation, then constructing one easy example per genus to check our property would settle the problem in full generality.

For further use in this introduction, let us distinguish between two kinds of deformation invariants.

Definition 2. In dealing with a class of objects A, let us call a function f on A to be a *deformation invariant* if $f(\xi) = f(\xi')$ when ξ can be deformed into ξ'. Let us call f a *complete deformation invariant* if, moreover, $\forall \xi, \xi' \in A$ the equality $f(\xi) = f(\xi')$ implies ξ can be deformed into ξ'.

The following theorem over the complex numbers is classical. It was later proved over any algebraically closed base field by Deligne and Mumford in [DM69].

Theorem 3. *The genus is a complete deformation invariant of curves*[1].

Unfortunately, for many problems, even a handpicked smooth genus g curve ends up being either too difficult to analyze or too special to say anything of value. The solution to this difficulty has been in use for over a century: deform the curve until it "snaps" into a *singular* curve and study the singular curve. A singular curve of genus g is built from smooth curves of genus strictly less than g. Therefore, after degeneration to a singular curve we can often use induction on genus to solve the problem, or even degenerate further until each piece of the curve is of genus 1 or 0; the simplest of curves.

Systematizing this approach by selecting the right set of singularities to allow in our deformations and putting them together into a single parametrizing space was one of the major features of the work of Deligne and Mumford [DM69]. The right objects to study were named *stable curves*. These are connected curves with at most nodal singularities and with finite automorphism groups. The result of Deligne and Mumford is in fact proved in this larger context: The genus is the only deformation invariant of a *stable* curve.

The value of their work then becomes clear: it provides the very framework in which the complexity of high genus curves can be systematically broken down.

1.1 Moduli spaces of curves

In modern language we can summarize the preceding discussion as follows. There exists a moduli space[2] $\mathcal{M}_g$ parametrizing smooth curves of genus g. Moreover, there exists a compactification $\overline{\mathcal{M}}_g$ of $\mathcal{M}_g$ via stable curves ([DM69]).

The fact that the Kodaira dimension of $\overline{\mathcal{M}}_g$ is non-negative in the range $g \geq 22$ ([HM82], [Har84] [EH87],[Far09]) implies that any map $\mathbb{P}^1 \to \overline{\mathcal{M}}_g$ with image containing a general point has to be constant. That is why we can not deform curves of high genus with a free parameter.

Our own work roughly adheres to this framework. Instead of the moduli space of curves, we consider the moduli space of *multiple spin curves* $\mathcal{S}_g^m$. In Part I of this thesis we provide a compactification $\overline{\mathcal{S}}_g^m$ of $\mathcal{S}_g^m$ so that we may attack problems through degeneration. These spaces are not irreducible, so we classify their irreducible components; in other words, we determine complete deformation invariants of multiple spin curves.

This was done primarily to address our guiding problems: the study of common contact curves defined in Section 4. In Part II of this thesis we turn to address this problem relying heavily on the newly constructed moduli spaces. As a by product, we get to say something about the Kodaira dimension of the related spaces.

Having introduced curves briefly, let us now turn to *spin* curves, which require the additional data of a theta characteristic. After that, we will be able to discuss our guiding problems and multiple spin curves in detail.

[1] More commonly, one would say that the moduli space of genus g curves is irreducible.

[2] We will not distinguish between fine and coarse moduli spaces for now.

2 Theta characteristics

Let C be a curve of genus g and ω_C its cotangent bundle. Let $V = \mathrm{H}^0(C, \omega_C)$ be the global sections of ω_C or, equivalently, the vector space of holomorphic differential forms on C. For each $p \in C$ let $V_p := \{\sigma \in V \mid \sigma(p) = 0\}$ be the subvector space of V consisting of all holomorphic differentials vanishing at p. Using Riemann–Roch we conclude V_p has codimension 1 in V for all $p \in C$. Let $\mathbb{P}(V)$ denote the space of codimension 1 subspaces of V, or equivalently the space of lines $|V^\vee|$ in the dual space.

Definition 4. The map $\varphi : C \to \mathbb{P}(V) : p \mapsto V_p$ is called the *canonical map.* If C is not hyperelliptic then φ is an embedding and $\varphi(C)$ is called the *canonical model of* C, or simply a *canonical curve.*

Since $\dim_{\mathbb{C}} V = g$ we have $\mathbb{P}(V) \simeq \mathbb{P}^{g-1}$. For the rest of this introduction, two examples will be sufficient to motivate the discussion to follow:

Example 5. Let $g = 3$ and C be non-hyperelliptic. Then $\varphi(C) \subset \mathbb{P}^2$ is a quadric plane curve. That is, there is a single homogeneous equation $f \in \mathbb{C}[x, y, z]$ of degree 4 such that $\varphi(C)$ is the locus of zeros of f. Conversely, every smooth plane quartic is a canonical curve of genus 3.

Example 6. If $g = 4$ and C is not hyperelliptic then $\varphi(C) \subset \mathbb{P}^3$ is the zero set of two polynomials, one of degree 2 and one of degree 3. Conversely, every smooth curve obtained by intersecting in $\mathbb{P}^3$ a quadric and a cubic is a canonical curve of genus 4.

Although treating curves in the abstract proves useful at times, there is something to be gained from an embedding. The extrinsic geometry of a curve in projective space gives a mental foothold on each curve: we can talk about the Gauss map, inflection points, bitangents, projections etc. All these can also be phrased in terms of objects intrinsic to the curve. But the language of projective geometry is highly suggestive and for that reason alone it is worth using.

Given any one curve C, there will be numerous ways to embed C into projective space. However, the moment we start deforming the curve, it turns out that the canonical map (and its multiples) is the only embedding which can also be deformed together with C. Roughly speaking, this is the content of Franchetta's conjecture (a theorem due to Harer [Har83] in the weak form, [Mes87] in the strong form and [Sch03] in a yet stronger form).

In other words, deformation theoretic questions regarding a general curve C can only be cast into extrinsic geometry through the canonical model. One particularly beautiful feature of the canonical curve is given by its theta hyperplanes, a subject which we turn to now.

2.1 Theta hyperplanes

Let C be a non-hyperelliptic genus g curve with $\varphi : C \to \mathbb{P}(V) \simeq \mathbb{P}^{g-1}$ the canonical embedding. Then for any hyperplane $H \subset \mathbb{P}(V)$ the intersection $\varphi^* H = H \cdot C$ gives a divisor of degree $2g - 2$ on C.

Definition 7. If $H \cdot C = 2D$ for a divisor D in C then H is called a *theta hyperplane.*

Remark 8. Consider the real parametrized curve $\gamma : \mathbb{R} \to \mathbb{R}^2 : t \mapsto (t, t^n)$ for some $n \geq 1$. The image of γ will pass through the x-axis whenever n is odd and it will touch the x-axis and move away if n is even. The distinction between even and odd contact which we observe with these real curves is in fact picked up by purely algebraic methods: the divisibility by 2 of the intersection divisor $H \cdot C$ above has significant consequences.

For $g = 3$, theta hyperplanes are the classical 28 bitangents of a plane quartic. This subject provided much of the impetus for the theory of theta hyperplanes in general. We refer to Chapter 6 of [Dol12] for more on this topic.

When $g = 4$, there are 120 theta hyperplanes which are often called tritangent planes. The literature here is far more scarce mainly because there is a large jump in complexity in passing from genus 3 to 4, see [Mil22] and [Cob82]. However, interest in the theta characteristics of low genus curves is growing because of the role they play in numerical computations, see [FK06] and [Shi86].

Before we say more about the theta hyperplanes, let us view them from a different perspective: as roots of the canonical bundle. Indeed, $\varphi^* H = D'$ means that D' is a canonical divisor, and so $\mathcal{O}_C(D') \simeq \omega_C$. If $\varphi^* H = 2D$ then $\mathcal{O}_C(D)$ is a line bundle with square ω_C, i.e., $\mathcal{O}_C(D)^{\otimes 2} \simeq \omega_C$. In modern guise, this is how theta characteristics appear:

Definition 9. If L is a line bundle on a curve C such that $L^{\otimes 2} \simeq \omega_C$ then L is called a *theta characteristic* of C.

If $D \in |L|$ is the zero divisor of a section of L then $2D \in |L^{\otimes 2}| = |\omega_C|$ is a canonical divisor so that there exists an $H \subset \mathbb{P}(V)$ with $\varphi^* H = 2D$. In other words, pairs (L, D) where L is a theta characteristic and $D \in |L|$ are in bijective correspondence with theta hyperplanes.

On a smooth curve C, the Jacobian of the curve, Jac_C, is a complex torus of dimension g and so a quotient of $\mathbb{C}^g$ by a lattice isomorphic to $\mathbb{Z}^{2g}$. Working with the Jacobian we see that there are 2^{2g} square roots of the trivial line bundle $\mathcal{O}_C$, this 2-torsion subgroup is denoted by $\mathrm{Jac}_C[2]$ which naturally has the structure of a $\mathbb{F}_2$ vector space of dimension g. Moreover, the Picard group $\mathrm{Pic}^d(C)$ of degree d line bundles on C is isomorphic as a complex manifold to Jac_C and the fibers of the squaring map $\mathrm{Pic}^d(C) \to \mathrm{Pic}^{2d}(C)$ must clearly be $\mathrm{Jac}_C[2]$-affine spaces. Then, by dimension reasons, the squaring map must be surjective. In other words, every line bundle of even degree has 2^{2g} square roots. In particular, there are 2^{2g} theta characteristics on a genus g curve.

We mentioned that the genus 3 curve has only 28 bitangents, not 64. This is because some theta characteristics do not admit global sections. In fact, the following is a classical result with a purely algebraic proof due to Mumford [Mum71] for the count and due to [Har82] for a precise generality statement.

Theorem 10. *On a genus g curve C there are $\binom{2^g}{2}$ theta characteristics η such that $h^0(\eta) \equiv 1 \pmod 2$ and $\binom{2^g+1}{2}$ theta characteristics η with $h^0(\eta) \equiv 0 \pmod 2$. If C is general then $h^0(\eta)$ is minimal for each theta characteristic, that is $h^0(\eta) \in \{0, 1\}$.*

Definition 11. A theta characteristic η will be called *even* or *odd* according to the parity of the integer $h^0(\eta)$.

Definition 12. If a theta characteristic η has the minimal number of possible sections, $h^0(\eta) \in \{0, 1\}$, then we will call η *rigid*.

Using Clifford's theorem we may conclude that on non-hyperelliptic genus 3 curves there are precisely 28 odd theta characteristics, each with one dimensional space of global sections, and 36 even theta characteristics each with no global sections.

If C is a curve of genus 4, necessarily all odd theta characteristics of C will have $h^0 = 1$ by Clifford's theorem and the even theta characteristics will have $h^0 \in \{0, 2\}$. With a little bit projective geometry, one can do better: If C is non-hyperelliptic, then the canonical model of C lies on a *smooth* quadric iff none of the even theta characteristics have any global sections. In this case, C will have precisely 120 tritangents. Otherwise, C lies on a singular quadric Λ and there are 120 tritangents that do not pass through the node of Λ and a pencil of tritangent hyperplanes which are precisely the tangent hyperplanes of Λ: each of these hyperplanes pass through the node of Λ and each is tangential to Λ along a different ray.

3 Deformations of theta characteristics

The physics nomenclature had an impact in the naming of curves taken together with a theta characteristic:

Definition 13. A tuple (C, L, α) where C is a curve of genus g and L is an odd (or even) theta characteristic with $\alpha : L^{\otimes 2} \xrightarrow{\sim} \omega_C$ is called an odd (or even) *spin curve*. If $h^0(L) \in \{0, 1\}$ we will say (C, L, α) is *rigid*.

Since there are only finitely many theta characteristics on a curve, deformations of theta characteristics are meaningless unless we deform the curve as well. Mumford [Mum71] was the first one to put this notion on an algebraic footing with the following definition:

Definition 14. If $\pi : \mathcal{X} \to B$ is a family of curves and ω_π is the relative cotangent bundle of π then a line bundle $\mathcal{L}$ on $\mathcal{X}$ such that $\mathcal{L}^{\otimes 2} \simeq \omega_\pi$ is a *family of theta characteristics*. If we fix $\alpha : \mathcal{L}^{\otimes 2} \simeq \omega_\pi$ then the triplet $(\pi, \mathcal{L}, \alpha)$ is called a *family of spin curves*.

For each $b \in B$, the relative cotangent bundle ω_π restricts to the cotangent bundle of the fiber $C_b := \pi^{-1}(b)$. Therefore, $\mathcal{L}$ will restrict to a theta characteristic on each fiber. The following result was first established by Atiyah and Mumford.

Theorem 15 ([Mum71],[Ati71])**.** *The parity of a spin curve of genus g is a deformation invariant.*

The following stronger result is claimed to be folklore in [Cor89]. However, the first algebraic proof of this fact is given there.

Theorem 16 ([Cor89]). *The parity of a spin curve of genus g is a* complete *deformation invariant.*

In other words, if we form the moduli space $\mathcal{S}_g$ of spin curves of genus g then the even and odd spin curves form irreducible, disjoint components $\mathcal{S}_g^-$ and $\mathcal{S}_g^+$ of $\mathcal{S}_g$ respectively. However, as before, it is of great advantage to be able to deform smooth spin curves into singular ones. This requires a compactification of $\mathcal{S}_g$, preferably one over $\overline{\mathcal{M}}_g$. This task was accomplished by Cornalba [Cor89], where he constructed a compactified moduli space $\overline{\mathcal{S}}_g = \overline{\mathcal{S}}_g^- \sqcup \overline{\mathcal{S}}_g^+$ using line bundles on so called *quasi-stable* curves.

The compactified spaces of Cornalba generated a lot of interest. In [CCC07], Cornalba's methods were generalized to r-th roots of line bundles and in [Jar98] Jarvis approached the problem of compactification using an equivalent but technically more advantageous approach building upon Faltings' idea [Fal96] of using torsion-free sheaves on stable curves.

Later, Ludwig in [Lud10] studied the singularities of $\overline{\mathcal{S}}_g$ opening the doors for a Kodaira dimension classification in the spirit of [HM82]. The Kodaira classification of $\overline{\mathcal{S}}_g^+$ for all $g \geq 2$ was realized by Farkas [Far10], the case $g = 8$ being completed in [FV12], and the Kodaira classification of $\overline{\mathcal{S}}_g^-$ for all $g \geq 2$ was realized by Farkas and Verra [FV14]. Compare this to the Kodaira classification of $\overline{\mathcal{M}}_g$ which is still open in the range $17 \leq g \leq 21$ and $g = 23$.

3.1 Syzygetic triplets

Take a smooth quadric $C \subset \mathbb{P}^2$ and 3 distinct bitangents ℓ_1, ℓ_2, ℓ_3. Let $p_{2i-1}, p_{2i} \in C$ be the two contact points of ℓ_i with C. Through 5 general points in $\mathbb{P}^2$ there passes a unique quadric and through 6 general points there will pass no quadrics. Admittedly the points $p_1, \dots, p_6$ are not random and so they need not be general with respect to quadrics. In fact, for each C and for about half of the triplets of bitangents (ℓ_1, ℓ_2, ℓ_3) there will be a quadric passing through the 6 contact points in which case the triplet (ℓ_1, ℓ_2, ℓ_3) will be called *syzygetic*.

What is perhaps more surprising, taken at face value, is the following. Given a syzygetic triplet with contact points $p_1, \dots, p_6$ and the quadric Q passing through them, we will have $Q \cdot C = p_1 + \cdots + p_8$ where $p_7, p_8 \in C$ are the remaining 2 points of the intersection. Then there exists yet another bitangent ℓ_4 such that $\ell_4 \cdot C = 2(p_7 + p_8)$.

In addition to this fascinating picture, the notion of a syzygetic triplet plays a fundamental role from the viewpoint of deformations. Although we suspect the following result must have been known as we state it now, in any case it follows from Theorem II.4.22:

Theorem 17. *The only deformation invariant of a genus 3 curve taken together with three distinct bitangents is whether or not the triplet is syzygetic.*

The notion of the syzygy of triplets of odd theta characteristics may be extended as follows: Given three distinct theta hyperplanes H_1, H_2, H_3 of a canonical curve $C \subset \mathbb{P}^{g-1}$, we say (H_1, H_2, H_3) is *syzygetic* if there exists a quadric which does not contain C and passes through the $3g-3$ points of contact of these three hyperplanes with the curve. They are called *asyzygetic* otherwise. Once again, the notion of syzygy turns out to be the only deformation invariant of triplets of odd theta characteristics.

Forming the moduli space of triple odd spin curves of genus g, this moduli space will consist of two disjoint irreducible components parametrizing those triplets which are syzygetic and those which are not.

Remark 18. Theorem II.4.22 considers the moduli space of curves with m-tuples of theta characteristics. The corresponding moduli space is denoted by $\mathcal{S}_g^{\times m}$ and we classify all the irreducible components according to the syzygy conditions of subtriplets.

4 Curves of common contact

Let $C \hookrightarrow \mathbb{P}^{g-1}$ be a canonical curve of genus g and $H \subset \mathbb{P}^{g-1}$ a theta hyperplane of C. The divisor D on C satisfying $2D = H \cdot C$ will be called a *contact divisor*. Any point $p \in C$ with $p \leq D$ will be called a *contact point*, or a *point of contact of H*.

Definition 19. If C admits two theta hyperplanes H_1 and H_2 having a common point of contact, then we will say C is a *common contact curve*.

This thesis is bent upon an investigation of common contact curves. The questions that led our investigation, stated informally, are as follows:

Problem A: How often do common contact curves appear in moduli?

Problem B: If H_1 and H_2 have at least one common contact point, will they have other common contact points?

Problem C: If C admits a pair of theta hyperplanes (H_1, H_2) having a common contact point, will C admit other such pairs?

Before we refine these questions further, let us investigate curves of low genus using basic tools to see how far we can go. In order to extend the usefulness of these basic tools and later to have a larger array of examples, it will be necessary to generalize the notion of common contact to hyperelliptic curves.

Recalling the equivalence between contact divisors and rigid odd theta characteristics, we will call $D \in |\eta|$ a *contact divisor* on any rigid odd spin curve (C, η). Accordingly we extend the definition of common contact points and common contact curves to possibly hyperelliptic curves.

4.1 Low genus with elementary methods

Genera 1 and 2

For genus 1 there is nothing to do, because an elliptic curve does not admit two *distinct* odd theta characteristics. In any case, the only odd theta characteristic is the trivial line bundle which has empty contact locus.

As for genus 2, each contact divisor is supported on exactly one Weierstrass point. In particular, no two distinct contact divisors can possibly share a point of contact.

Genus 3

If C is a genus 3 non-hyperelliptic curve then its odd theta characteristics correspond to the bitangents of the canonical model of C, which is a plane quartic. Any bitangent is determined by any one of its point of contact: it is the tangent line at that point. For this reason, no two distinct odd theta characteristics have a common contact point.

Suppose C is a genus 3 hyperelliptic curve and let $W = \{w_1, \dots, w_8\}$ be its Weierstrass points. Any odd theta characteristic on C is of the form $\mathcal{O}_C(w_i + w_j)$ for $i \neq j$ (see Chapter III.1). This gives 336 ordered pairs of odd theta characteristics with a common point of contact.

Note that, hyperelliptic genus 3 curves form a divisor in $\mathcal{M}_3$. Therefore, we see that the locus of common contact curves are divisorial in $\mathcal{M}_3$.

Genus 4

Let $C \hookrightarrow \mathbb{P}^3$ be a canonical genus 4 curve. We will show that no two tritangent planes on C can have 2 common contact points.

The idea is to show that the tangent directions of two points span at least a plane. This would imply that any plane tangent to two distinct points should be uniquely determined by those two points. To verify this statement we need to prove that C has no bitangent lines.

Suppose to the contrary that a line through $p, q \in C$ is a bitangent. Then $h^0(K_C - 2p - 2q) = 2$ by geometric Riemann–Roch. But this is impossible since C is not hyperelliptic and $\deg(K_C - 2p - 2q) = 2$. Of course, a hyperelliptic curve of genus 4 does admit 2 common contact points for certain pairs of rigid odd theta characteristics.

High genus with elementary methods

The argument used for $g = 3, 4$ can be applied to all genera $g \geq 3$, though with diminishing returns. We will see later that the generic curve of any genus will not admit any contact points at all, so the following bound grows progressively worse.

Proposition 20. *Let $g \geq 3$ and C be a Brill-Noether general smooth curve of genus g. If H_1 and H_2 are two distinct contact hyperplanes of C then there are at most $\lceil \frac{3(g-3)}{4} \rceil$ common contact points of the pair (H_1, H_2).*

Proof. Suppose there are n common contact points of H_1 and H_2. Denote them by $p_1, \dots, p_n$. Note that a contact hyperplane has at most $g - 1$ contact points, so it is convenient to write $n = g - 1 - i$ for some $i > 0$.

By hypothesis, the linear series $|K_C - 2p_1 - \cdots - 2p_n|$ contains at least two distinct divisors, namely $H_j \cap C$ for $j = 1, 2$. Thus C has a $g^1_{2(g-1-n)} = g^1_{2i}$. Assuming C is general, the Brill-Noether formula $\rho(g,1,2i)\geq 0$ bounds i giving us the desired bound for n above. $\square$

4.2 Progress on the driving problems

In order to make our problems precise, consider the moduli space of double odd spin curves

$$\mathcal{S}_g^{--}=\{(C,\eta_1,\eta_2)\mid h^0(\eta_i)\equiv 1 \pmod 2),\ \eta_1\not\cong\eta_2\}.$$

Outside of a codimension 3 locus in $\mathcal{S}_g^{--}$, odd theta characteristics are rigid (see [Har82]) so for the rest of this section we will assume $h^0(\eta_i)=1$ without further comment.

Define the locus $\Omega_g\subset\mathcal{S}_g^{--}$ of double spin curves (C,η_1,η_2) with η_1 and η_2 having a common contact point. We prove in Chapter II.2 that Ω_g is pure of codimension 1 in $\mathcal{S}_g^{--}$. In particular, a general one parameter family of double spin curves will intersect Ω_g in finitely many points. Thus we refine Problem A:

Problem A′: At how many points does a one parameter family of double spin curves intersect Ω_g?

An intersection theoretic question of this nature behaves much better in a compact ambient space. Therefore, we instead use the compactified moduli space $\overline{\mathcal{S}}_g^{--}$ which we define in Part I and study in detail in Part II. We then take the Zariski closure $\overline{\Omega}_g$ of Ω_g in $\overline{\mathcal{S}}_g^{--}$.

In order to answer Problem A′ we need to express the divisor class $[\overline{\Omega}_g]\in\mathrm{Pic}_{\mathbb{Q}}(\overline{\mathcal{S}}_g^{--})$ in terms other, simpler, classes which we introduce now.

The complement $\overline{\mathcal{S}}_g^{--}\setminus\mathcal{S}_g^{--}$ is the union of *boundary divisors* which we denote by Δ_i^{xy}, see Chapter II.1 for a comprehensive treatment. We will denote by δ_i^{xy} the divisor class $[\Delta_i^{xy}]$. In addition, there is the Hodge class $\lambda\in\mathrm{Pic}_{\mathbb{Q}}(\overline{\mathcal{S}}_g^{--})$ which essentially measures how much the ambient space $\mathbb{P}^{g-1}$ has to twist in order to receive the canonical map from a given family of curves.

Theorem 21 (see Corollary II.2.45)**.** *If $\overline{\Omega}_g$ is irreducible then we have:*

$$\begin{aligned}[\overline{\Omega}_g]=\frac{g+5}{2}\lambda-\frac{g+1}{8}\delta_0^{nn}-\frac{g+3}{8}\left(\delta_0^{nb}+\delta_0^{bn}\right)-\delta_0^{bb}-(g-1)\delta_0^{b=}\\ -\sum_{i=1}^{g-1}\left((2i-1)\delta_i^{++}-(g-1)\delta_i^{+-}-(3i-1)\delta_i^{+=}-(g+i-2)\delta_{g-i}^{-=}\right)\end{aligned}$$

Remark 22. See Theorem II.2.43 for an unconditional statement.

Remark 23. In Section 4.1 above we were able to give an explicit description of Ω_3 in terms of hyperelliptic curves and pairs of Weierstrass points. Using this interpretation, one can completely describe the Zariski closure $\overline{\Omega}_3$ using the theory of admissible covers (see [HM98]). Moreover, our discussion of the boundary components in Section II.1.7 allows us to build test curves similar to the test curves used in $\overline{\mathcal{M}}_g$ (see [HM82]). Combining these together provides the means for a completely independent verification of the formula above in $g = 3$. We carried out these computations and the two results agree.

As a byproduct of this computation, but unconditional on the irreducibility of $\overline{\Omega}_g$, we prove the following:

Theorem 24 (see Section II.2.6)**.** *The coarse moduli scheme $\overline{S}_g^{--}$ of $\overline{\mathcal{S}}_g^{--}$ has big canonical class when $g \geq 10$. In particular, if $\overline{S}_g^{--}$ has mild singularities then it is of maximal Kodaira dimension when $g \geq 10$.*

Problem B is easy to to answer with the hypothesis that $\overline{\Omega}_g$ is irreducible. First, notice that this problem can be rephrased as follows:

Problem B′: If we take a generic element $(C, \eta_1, \eta_2) \in \Omega_g$ then how many points of common contact will the η_i's have?

Proposition 25. *If $\overline{\Omega}_g$ is irreducible and $(C, \eta_1, \eta_2) \in \overline{\Omega}_g$ is generic, then $\#(\eta_1 \cap \eta_2) = 1$.*

Proof. Degenerate to a hyperelliptic curve C with Weierstrass points $\{w_1, \ldots, w_{2g+2}\}$. Choose $\eta_1 = \sum_{i=1}^{g-1} w_i$ and $\eta_2 = \sum_{i=g-1}^{2g} w_i$. Then $h^0(\eta_i) = 1$ and $\#(\eta_1 \cap \eta_2) = 1$. Application of semi-continuity theorem finishes the proof. See Chapter II.2 for a description of the space of contact points over Ω_g. □

Remark 26. Note that our discussion in Section 4.1 implies that for *any* $(C, \eta_1, \eta_2) \in \Omega_4$ we must have $\#(\eta_1 \cap \eta_2) = 1$, independent of irreducibility of $\overline{\Omega}_4$.

The guiding Problem C turns out to be much harder. We will outline our progress here. Let us first recast Problem C into our current notation.

Problem C′: What is the degree of $\overline{\Omega}_g$ over its image in $\overline{\mathcal{M}}_g$?

Note that this degree counts each pair of hyperplanes twice due to the ordering we imposed on the pair of odd theta characteristics in defining the moduli space $\mathcal{S}_g^{--}$.

Recall that when $g = 3$, the locus Ω_3 will have degree 336 over its image. However, this appears to be an exceptional case and we expect the degree to be 2 in the range $g \geq 4$.

Let $\pi : \overline{\mathcal{S}}_g^{--} \to \overline{\mathcal{M}}_g$ be the map forgetting the spin structures. For any $X \in \overline{\mathcal{M}}_g$ let f_X be the number of points in the intersection $\pi^{-1}(X) \cap \overline{\Omega}_g$. Since $\overline{\Omega}_g$ is divisorial, for generic X we will have $f_X = 0$. Assuming $\overline{\Omega}_g$ is irreducible, if we find one X for which $f_X = 2$ then by semi-continuity we would conclude $\overline{\Omega}_g$ is of degree 2 over its image.

We find good candidates in the locus of irreducible nodal curves $\Delta_0 \subset \overline{\mathcal{M}}_g$. We show in Section II.1.7.2 that $\pi^{-1}(\Delta_0)$ breaks into five irreducible components: Δ_0^{bb}, $\Delta_0^{b=}$, Δ_0^{bn}, Δ_0^{nb} and Δ_0^{nn}. Therefore, for $X \in \Delta_0$, we can break the computation of f_X into five pieces. Simply define $f_X^{xy} := \#(\pi^{-1}(X) \cap \overline{\Omega}_g \cap \Delta_0^{xy})$ so that $f_X = f_X^{bb} + f_X^{b=} + f_X^{bn} + f_X^{nb} + f_X^{nn}$.

We obtain the following result:

Theorem (see Theorem II.3.2). *When $g \geq 4$ there exists a curve $X \in \Delta_0$ such that $f_X^{bb} = 2$ and $f_X^{b=} = f_X^{bn} = f_X^{nb} = 0$.*

This leaves the computation of f_X^{nn}, which we are unable to do. However, we compute it conditionally. See Remark II.3.4 for some evidence on why we think the condition must hold.

Lemma (see Corollary II.3.5). *For the same X, we have $f_X^{nn} = 0$ provided that there exists a hyperelliptic curve C of genus $g-1$ and two Weierstrass points $w_1, w_2 \in C$ such that any distinct pair of roots $\tau_1, \tau_2 \in \sqrt{\omega_C(w_1 + w_2)}$ has disjoint zero divisors.*

Corollary 27. *If $\overline{\Omega}_g$ is irreducible and the hypothesis of the lemma above is satisfied then $\overline{\Omega}_g$ is of degree 2 over its image in $\overline{\mathcal{M}}_g$.*

For emphasis on why we are interested in this result, we summarize the implications below.

Corollary 28. *If the hypotheses of Corollary 27 hold, then the generic common contact curve admits a unique unordered pair of theta hyperplanes having common contact. In light of Proposition 25, this pair will have a unique common contact point.*

4.3 Irreducibility of $\overline{\Omega}_g$

We conjecture that $\overline{\Omega}_g$ is irreducible. This is easily seen to be true when $g = 3$ by the monodromy action of the covering $\Omega_3 \to \mathcal{H}_3$, see Chapter II.3 for more detailed arguments of this sort. When $g \geq 4$ we make the following observations.

We classify the intersection of $\overline{\Omega}_g$ with the boundary $\overline{\mathcal{S}}_g^{--} \backslash \mathcal{S}_g^{--}$. Using this classification and Theorem III.1.8 we may conclude that the intersection $\overline{\Omega}_g \cap \Delta_0^{bb}$ is irreducible. Therefore, exactly one component of $\overline{\Omega}_g$ can intersect Δ_0^{bb}. We know that in $\overline{\mathcal{M}}_g$, any effective divisor intersects every one of the boundary components, see [Fab89]. Therefore, we may speculate that every component of $\overline{\Omega}_g$ will intersect Δ_0^{bb} implying the irreducibility of $\overline{\Omega}_g$.

One could also use Faber's observation directly, but with greater difficulty. Since $\pi(\overline{\Omega}_g)$ must intersect Δ_1, $\overline{\Omega}_g$ must intersect $\pi^{-1}(\Delta_1)$. We classify the intersection loci of $\overline{\Omega}_g$ with every component of $\pi^{-1}(\Delta_1)$ in Chapter II.3. For instance, the intersection of $\overline{\Omega}_g$ with $\Delta_1^{+-} \subset \pi^{-1}(\Delta_1)$ breaks into two. One of them is essentially Ω_{g-1} and the other is essentially Ω_{g-1}^{+}, the locus of Scorza points (see Definition II.3.43). We may assume Ω_{g-1} is irreducible by induction and we prove in Corollary II.3.50 that Ω_g^{+} is irreducible. Finally, it is easy to construct an irreducible family of hyperelliptic curves in Ω_g having limit points in both of these limit components Ω_{g-1} and Ω_{g-1}^{+}. Similarly, one can connect the other intersection loci in $\pi^{-1}(\Delta_1)$ to one another. This would imply the irreducibility of $\overline{\Omega}_g$, unless the hyperelliptic locus appears in the intersection of two components of $\overline{\Omega}_g$.

Despite the gap in both attempts, we feel there is compelling evidence to suggest that $\overline{\Omega}_g$ is irreducible. We believe Part II of this thesis should provide a strong foundation to attack this problem. Nevertheless, even a direct application of one of the two strategies outlined above would seem to require a good deal of insight.

5 Compactifying the moduli of multiple spin curves

Let C be a proper smooth curve with a line bundle N. Consider an m-tuple of square roots of N, that is, a sequence of line bundles $L_1, \ldots, L_m$ such that $L_i^{\otimes 2} \simeq N$.

Given a degeneration of (C, N) to a singular stable curve, it is known how each root L_i will deform. However, if these individual degenerations are allowed to degenerate independently then we do not get a satisfactory theory for the degeneration of m-tuples of roots. More precisely, the associated moduli stack is non-normal and the underlying degenerations are unnatural.

In Part I we describe how to synchronize these degenerations and prove that the resulting moduli problem is represented by a smooth Deligne–Mumford stack. Moreover, we show that the resulting degenerations are geometrically meaningful.

In the paper [FJR13] a compactification of the moduli of various arrangements of roots of the canonical bundle is constructed using line bundles on twisted curves. We adopt the more geometric point of view and use line bundles on quasi-stable curves, or equivalently torsion-free sheaves on stable curves, to compactify our moduli space.

5.1 Statement of the result for multiple spin curves

The initial goal of Part I has been to find a "good" compactification for the m-fold product $\mathcal{S}_g^{\times m} = \mathcal{S}_g \times_{\mathcal{M}_g} \cdots \times_{\mathcal{M}_g} \mathcal{S}_g$, i.e., the moduli space of curves with an m-tuple of spin structures. We achieved this goal in greater generality, not restricting ourselves to the roots of the canonical bundle. For now however, we will describe our main result for this specific case.

Let us point out that the obvious compactification $\overline{\mathcal{S}}_g^{\times m} = \overline{\mathcal{S}}_g \times_{\overline{\mathcal{M}}_g} \cdots \times_{\overline{\mathcal{M}}_g} \overline{\mathcal{S}}_g$ is non-normal (see Section I.5.5). There is another problem with this compactification: the objects it parametrizes are unnatural as we will see below.

The moduli space $\overline{\mathcal{S}}_g$ parametrizes limit spin curves. These are triplets $(X, L, \alpha : L^{\otimes 2} \to \omega_X)$ where X is a *quasi*-stable curve (Definition III.2.2), L is a line bundle on X and α is almost an isomorphism (Definition III.2.7). The forgetful map $\overline{\mathcal{S}}_g \to \overline{\mathcal{M}}_g$ sends (X, L, α) to the stabilization C of X.

When we consider the product $\overline{\mathcal{S}}_g^{\times m}$, the objects we parametrize would then be m-tuples of the form $(\pi_i : X_i \to C, L_i, \alpha_i)_{i=1}^m$ where each π_i is the stabilization map. In other words, the stabilizations are identified but not the quasi-stable curves X_i. So we end up with m line bundles on m different curves!

A good compactification of $\mathcal{S}_g^{\times m}$ should parametrize objects that are of the form $(X, \{L_i, \alpha_i\}_{i=1}^m)$ where X is quasi-stable and each (X, L_i, α_i) is a limit spin curve, possibly after a partial stabilization of X.

If we leave it at that, our moduli space would not have finite fibers over $\overline{\mathcal{M}}_g$. To overcome this problem, we require that for each i, j the line bundles $L_i^{\otimes 2}$ and $L_j^{\otimes 2}$ are isomorphic around the unstable components on which they have the same degree (see Definition III.2.19).

Let us denote the resulting moduli space by $\overline{\mathcal{S}}_g^m$. Then our main result becomes:

Theorem 29. *The moduli space $\overline{\mathcal{S}}_g^m$ is proper and the inclusion $\mathcal{S}_g^{\times m} \hookrightarrow \overline{\mathcal{S}}_g^m$ is dense and open. The forgetful map $\overline{\mathcal{S}}_g^m \to \overline{\mathcal{M}}_g$ induces a finite map over the coarse moduli spaces. Furthermore, the stack $\overline{\mathcal{S}}_g^m$ is smooth.*

The objects parametrized by $\overline{\mathcal{S}}_g^m$ are built to be used for enumerative problems. Therefore the last condition, giving us the smoothness of $\overline{\mathcal{S}}_g^m$, is particularly valuable.

Remark 30. To phrase our main result precisely and in appropriate generality, we have to introduce quite a bit of technical machinery. We will do this in the introduction to Part I.

We are able to give a complete classification of the components of $\mathcal{S}_g^m$ and, therefore, of $\overline{\mathcal{S}}_g^m$. First, note that there are some components of $\mathcal{S}_g^m$, such as the diagonals, which truly require less than m spin structures to define. These, we will call *degenerate* components (see Definition II.4.15 for a precise definition).

The non-degenerate components of $\mathcal{S}_g^m$ are classified by the syzygy relations of the theta characteristics. Given $(C, \eta_1, \ldots, \eta_m) \in \mathcal{S}_g^m$ define $a_i \in \mathbb{F}_2$ such that $a_i = h^0(\eta_i) \pmod 2$. For all $i \neq j \in \{2, \ldots, m\}$ let $a_{ij} \in \mathbb{F}_2$ be such that $a_{ij} = h^0(\eta_i \otimes \eta_j \otimes \eta_1^\vee) \pmod 2$.

Theorem 31. *The element $\mathfrak{a} = (a_i; a_{kl}) \in \mathbb{F}_2^{m+\binom{m-1}{2}}$ is a complete deformation invariant of $(C, \eta_1, \ldots, \eta_m)$.*

More precisely, Lemma II.4.18 and Theorem II.4.22 states that when $m \leq g$ the irreducible non-degenerate components of $\mathcal{S}_g^m$ are in bijection with the elements $\mathfrak{a}$ of $\mathbb{F}_2^{m+\binom{m-1}{2}}$. When $g < m$ there will be fewer components and we provide in Remark II.4.23 an effective algorithm to check which of the expected components are empty.

Part I

Moduli spaces of multiple spin curves

Chapter 1

Introduction

1.1 Conventions

As we are going to work with m-tuples of roots, fix once and for all an integer $m \geq 1$.

Definition 1.1. Let $\mathcal{M}$ be an algebraic stack. Then, a *stable curve* over $\mathcal{M}$ is a proper, flat, finitely presented morphism $\pi : \mathcal{C} \to \mathcal{M}$ whose geometric fibers are reduced, connected and of dimension 1, with at worst nodal singularities and such that the relative dualizing sheaf $\omega_{\mathcal{C}/\mathcal{M}}$ is relatively ample.

It is well known that the moduli space of line bundles on a nodal curve is not proper. One way to compactify this space is via torsion-free sheaves of rank-1. We do not need this result at the moment but this fact motivates the following definition.

Definition 1.2 (Jarvis)**.** A *torsion-free sheaf on a stable curve* $\mathcal{C} \to \mathcal{M}$ is a coherent $\mathcal{O}_{\mathcal{C}}$-module $\mathcal{E}$ which is flat and of finite presentation over $\mathcal{M}$ such that over each $s \in \mathcal{M}$ the fiber $\mathcal{E}|_{\mathcal{C}_s}$ has no associated primes of height one.

In light of the fact that a line bundle is torsion-free, the following definition is an elegant generalization of the usual notion of a square root of a line bundle:

Definition 1.3 (Deligne, Jarvis)**.** Let $\mathcal{E}$ be a rank-1 torsion-free sheaf on a curve $\mathcal{C} \to \mathcal{M}$ and $\mathcal{N}$ a line bundle on $\mathcal{C}$. Let $\delta : \mathcal{E} \xrightarrow{\sim} \mathcal{N} \otimes \mathcal{E}^\vee$ be an isomorphism. Then the pair $(\mathcal{E}, \delta)$ will be called a *(square) root of* $\mathcal{N}$.

Definition 1.4. Given a coherent module $\mathcal{E}$ and a line bundle $\mathcal{N}$ on a scheme $\mathcal{X}$, a homomorphism $b : \mathcal{E}^{\otimes 2} \to \mathcal{N}$ will be called a *bilinear form*, with $\mathcal{N}$ understood from context.

Notice that a bilinear form induces two maps $b^l, b^r : \mathcal{E} \to \mathcal{E}^\vee \otimes \mathcal{N}$ where $\mathcal{E}^\vee = \hom(\mathcal{E}, \mathcal{O}_{\mathcal{X}})$, $b^r(e) = b(e, \_)$ and $b^l(e) = b(\_, e)$.

Definition 1.5. Given a bilinear form $b : \mathcal{E}^{\otimes 2} \to \mathcal{N}$, if both b^r and b^l are isomorphisms then b is said to be *non-degenerate*. If $b^r = b^l$ then b is *symmetric*, and then b factors through the symmetrizing map $\mathcal{E}^{\otimes 2} \to \operatorname{Sym}^2 \mathcal{E}$.

Notation for symmetric powers. We will adopt an unusual notational custom and for any A-module E write the d-th symmetric product $\mathrm{Sym}^d_A(E)$ simply as E^d, and given $\mu : E \to F$ we will denote by μ^d the induced map $E^d \to F^d$. The same goes for sheaves of modules and morphisms between them. In compensation, we will write out tensor powers and direct sums explicitly as $E^{\otimes d}$ and $E^{\oplus d}$, respectively.

We will now state our working definition of a root, which is equivalent to the definition of Deligne and Jarvis above.

Definition 1.6. Let $\mathcal{E}$ be a rank-1 torsion-free sheaf on a curve $\mathcal{C} \to \mathcal{M}$ and $\mathcal{N}$ a line bundle on $\mathcal{C}$. Let $b : \mathcal{E}^2 \to \mathcal{N}$ be a non-degenerate symmetric form. Then the pair $(\mathcal{E}, b)$ will be called a *(square) root of* $\mathcal{N}$ *on* $\mathcal{C}/\mathcal{M}$.

Remark 1.7. To see that Definition 1.6 is equivalent to Definition 1.3 proceed as follows. Given $\delta : \mathcal{E} \xrightarrow{\sim} \mathcal{N} \otimes \mathcal{E}^\vee$ we obtain a non-degenerate bilinear form $b : \mathcal{E}^{\otimes 2} \to \mathcal{N}$. We will prove in Remark 2.8 that any such b is in fact symmetric, giving a non-degenerate $b : \mathcal{E}^2 \to \mathcal{N}$. For the converse, given $(\mathcal{E}, b)$ let $\delta := b^r = b^l$.

Remark 1.8. Jarvis in [Jar98] works with non-degenerate forms $\mathcal{E}^{\otimes 2} \to \mathcal{N}$, although they are automatically symmetric. This is no problem when working with a single root. In considering tuples of roots however, carrying around the kernel of $\mathcal{E}^{\otimes 2} \to \mathcal{E}^2$ is disruptive and so we consider the equivalent formulation of roots using symmetric powers.

Remark 1.9. If we let $V \hookrightarrow \mathcal{C}$ denote the open locus on which $\mathcal{E}$ is free, then the smooth locus of the map $\mathcal{C} \to \mathcal{M}$ is contained in V. In addition, b is an isomorphism on V.

Definition 1.10. An isomorphism $\mu : (\mathcal{E}, b) \to (\mathcal{E}', b')$ of roots is defined to be an isomorphism of the underlying sheaf of modules $\mu : \mathcal{E} \xrightarrow{\sim} \mathcal{E}'$ such that $b = b' \circ \mu^2$.

Notation. By a DM stack we will mean a Deligne–Mumford stack.

1.2 Setting up the problem

Fix an excellent base scheme S defined over $\mathbb{Z}[1/2]$ (e.g., $S = \mathbb{Z}[1/2]$ or $S = \mathbb{C}$ will do) and let $\mathcal{M} \to S$ be a DM stack, locally of finite type over S.

Fix a stable curve $\mathcal{C} \to \mathcal{M}$ of genus $g \geq 2$, which need not be generically smooth. In addition, fix a line bundle $\mathcal{N}$ on $\mathcal{C}$ having absolutely bounded degree (see Definition 1.12). This is a very weak condition as we explain in Remark 1.13. However it is also a very useful one, since a line bundle with absolutely bounded degree can be twisted by a sufficiently high power of a relatively ample bundle, such as $\omega_{\mathcal{C}/\mathcal{M}}$, in order to kill the relative cohomology.

Given any $T \to \mathcal{M}$ we can pullback the curve $\mathcal{C}$ and $\mathcal{N}$ to get a stable curve $\mathcal{C}_T \to T$ together with a line bundle $\mathcal{N}_T$.

Definition 1.11. Let $\overline{\mathcal{S}}(\mathcal{N}) \to \mathcal{M}$ be the category fibered in groupoids whose objects over $T \to \mathcal{M}$ are roots of $\mathcal{N}_T$. Similarly, let $\mathcal{S}(\mathcal{N}) \subset \overline{\mathcal{S}}(\mathcal{N})$ be the subcategory consisting of roots that are locally free.

The arguments in [Jar98] imply that $\overline{\mathcal{S}}(\mathcal{N}) \to \mathcal{M}$ is an algebraic space (see Section 5.1.1), which compactifies $\mathcal{S}(\mathcal{N}) \to \mathcal{M}$. In fact, [Jar98] deals in a slightly more restricted setting where $\mathcal{C} \to \mathcal{M}$ is the universal curve over the moduli space of stable curves of genus $g \geq 2$. We replaced this with the boundedness condition on $\mathcal{N}$, which seems to be the key in establishing the algebraicity of $\overline{\mathcal{S}}(\mathcal{N})$.

Moreover, we will show in Section 5.1 that the arguments presented in *loc.cit.* imply that $\overline{\mathcal{S}}(\mathcal{N}) \to S$ is a Deligne–Mumford stack. Moreover, we show that $\overline{\mathcal{S}}(\mathcal{N}) \to S$ is smooth if $\mathcal{M} \to \overline{\mathcal{M}}_g$ is smooth.

Denote by $\mathcal{S}^m(\mathcal{N})$ the m-fold product $\mathcal{S}(\mathcal{N}) \times_{\mathcal{M}} \cdots \times_{\mathcal{M}} \mathcal{S}(\mathcal{N})$. Our goal is to find a "good" compactification of $\mathcal{S}^m(\mathcal{N})$. Something which $\overline{\mathcal{S}}(\mathcal{N}) \times_{\mathcal{M}} \cdots \times_{\mathcal{M}} \overline{\mathcal{S}}(\mathcal{N})$ fails to do, as it is non-normal (see Section 5.5) and the objects parametrized by this fiber product are not geometrically meaningful (see Proposition III.2.16).

The reader interested only in spin curves may simply take $\mathcal{M}$ to be the moduli space of stable curves of genus g, $\mathcal{C} \to \mathcal{M}$ to be the universal curve over it and $\mathcal{N}$ to be the relative dualizing sheaf $\omega_{\mathcal{C}/\mathcal{M}}$.

1.2.1 Roots of higher degree and twisted curves

In the literature, r-th roots of line bundles have already been studied: from the perspective of torsion-free sheaves in [Jar98] and from the equivalent perspective of quasi-stable curves in [CCC07]. We will only consider m-tuples of *square* roots ($r = 2$) because in passing to $r \geq 3$ a hefty technical price has to be paid even in defining the roots. We avoided this because we feel the theory of twisted curves is better suited to handle the theory of roots when $r \geq 3$.

As we mentioned in the beginning, a compactification of such tuples (of r-th roots of the canonical bundle, and its variations) is already constructed in [FJR13] using line bundles on twisted curves.

Nevertheless, the definition of *square* roots in terms of torsion-free sheaves is much shorter and far more accessible for geometric problems than twisted curves. With that said, we hope our current pursuit is well justified.

1.2.2 Absolutely bounded degree

Definition 1.12. If there exists a constant $c \in \mathbb{Z}$ such that on any component Y of any geometric fiber of $\mathcal{C} \to \mathcal{M}$ we have $\deg \mathcal{N}|_Y \geq c$ then $\mathcal{N}$ will be said to have *absolutely bounded degree.*

Remark 1.13. This boundedness condition is weak enough that unless $\mathcal{M} \to \overline{\mathcal{M}}_g$ has geometric fibers with infinitely many connected components, the condition is automatically satisfied. In any case, if $\mathcal{N} = \omega_{\mathcal{C}/\mathcal{M}}^{\otimes l}$ for any $l \in \mathbb{Z}$, then $\mathcal{N}$ has absolutely bounded degree (see Sublemma 4.1.10 [Jar98] for $l = 1$, the idea readily generalizes to all $l \in \mathbb{Z}$).

1.2.3 A remark about Artin's approximation theorem

Some of the results cited throughout were written when Artin's approximation theorem was known to be applicable only over a restricted class of excellent rings. Since then this restriction has been lifted (see [CdJ02]) and we will freely use the cited results over arbitrary excellent rings.

1.3 Statement of the result

We will define $\overline{\mathcal{S}}^m(\mathcal{N}) \to \mathcal{M}$ in Section 5.2 and prove the following:

Theorem 1.14. *$\overline{\mathcal{S}}^m(\mathcal{N})$ is a DM stack, locally of finite type, proper and quasi-finite over $\mathcal{M}$.*

Proof. See Theorem 5.29, Corollary 5.47 and Section 5.4. □

With further assumptions on $\mathcal{M}$ we can also say more about $\overline{\mathcal{S}}^m(\mathcal{N})$. Here are the most useful ones:

Theorem 1.15. *If $\mathcal{M} \to \overline{\mathcal{M}}_g$ is smooth then so is $\overline{\mathcal{S}}^m(\mathcal{N}) \to S$.*

Proof. This is Theorem 5.48. □

Theorem 1.16. *If $\mathcal{C} \to \mathcal{M}$ is generically smooth $\mathcal{S}^m(\mathcal{N}) \hookrightarrow \overline{\mathcal{S}}^m(\mathcal{N})$ is a dense open immersion.*

Proof. This is Corollary 5.50. □

Possibly the most studied setting is when $\mathcal{M} = \overline{\mathcal{M}}_{g,n}$ and when $\mathcal{C} = \overline{\mathcal{C}}_{g,n} \to \mathcal{M}$ is the universal curve. Denote by $\sigma_1, \dots \sigma_n : \overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{C}}_{g,n}$ the n markings. The hypothesis of the theorems above are satisfied and we have the following corollary.

Corollary 1.17. *For any $a_1, \dots, a_n \in \mathbb{Z}$ let*

$$\mathcal{N} = \mathcal{O}_{\overline{\mathcal{C}}_{g,n}}(\sum_{i=1}^{n} a_i \sigma_i) \quad \text{or} \quad \mathcal{N} = \omega_{\overline{\mathcal{C}}_{g,n}/\overline{\mathcal{M}}_{g,n}}(\sum_{i=1}^{n} a_i \sigma_i).$$

Then $\overline{\mathcal{S}}^m(\mathcal{N}) \to S$ is a proper smooth Deligne–Mumford stack over $\mathcal{M}$ and $\mathcal{S}^m(\mathcal{N}) \hookrightarrow \overline{\mathcal{S}}^m(\mathcal{N})$ is a dense open immersion.

This implies in particular that the coarse moduli space of $\overline{\mathcal{S}}^m(\mathcal{N})$ exists, is finite over the coarse moduli of $\overline{\mathcal{M}}_{g,n}$ and is projective over S (see Proposition 5.55).

This corollary agrees with the results of [FJR13] when $m \geq 2$ and with [Jar98] when $m = 1$.

Remark 1.18. These results may be more interesting for some when phrased in the language of limit roots and quasi-stable curves. For this reason in Chapter III.2 we make the equivalence between limit roots and (torsion-free) roots explicit.

1.4 Overview

In Chapter 2 we concentrate on the formal neighbourhood of the node of a curve over an algebraically closed field and describe how a root degenerates together with the node. This section is largely expository and is included for easy reference of technical lemmas.

In Chapter 3 we define how to "synchronize" the deformation of a sequence of m roots at a node. We then study the deformation of synchronized roots together with the node.

In Chapter 4 we bring together the results of the past two sections to study the deformations of a curve together with an m-tuple of synchronized roots. This section provides us with the local description of $\overline{\mathcal{S}}^m(\mathcal{N})$.

Finally, in Chapter 5 we define $\overline{\mathcal{S}}^m(\mathcal{N})$ in full generality and then prove that $\overline{\mathcal{S}}^m(\mathcal{N})$ is a DM stack. We end the section by establishing various properties of $\overline{\mathcal{S}}^m(\mathcal{N})$ such as being smooth and proper over S provided that $\mathcal{M}$ is reasonably nice. In Section 5.5 we prove that the product $\overline{\mathcal{S}}(\mathcal{N}) \times_{\mathcal{M}} \overline{\mathcal{S}}(\mathcal{N})$ will, in general, be non-normal.

See Chapter III.2 for another, more geometric, interpretation of synchronized m-tuples of roots in terms of line bundles on blow-ups of curves in the same vein as [Cor89] and [CCC07].

Chapter 2

Universal deformation of a node with a root

In this section we define the deformation of a node together with a root of a line bundle and then give the universal deformation. This amounts to bringing together the results available in the literature, i.e., in [Fal96] and [Jar98].

Faltings' paper studies torsion-free sheaves of finite rank, also with a non-degenerate quadratic form. Jarvis' paper studies rank-1 torsion-free sheaves as r-th roots of line bundles. However, rank-1 torsion-free sheaves considered as a square root (i.e., $r = 2$) lies in the intersection of these two papers and are by far the simplest to consider. Therefore, a treatment of this special case is quite revealing.

In addition, we provide a more detailed proof of Proposition 5.4.3 in [Jar98] for square roots. We package this result in Theorem 2.30.

2.1 Conventions

In this section and the next we will be concerned about (infinitesimal) deformations of an affine scheme, as these are always affine it is convenient to work in the dual category of algebras instead of schemes. However, the arrows are mostly written so that when we apply Spec to the diagrams they look familiar.

2.1.1 Notation

- k is an algebraically closed field of characteristic $\neq 2$.
- Λ is a complete noetherian local ring with residue field k.
- Art_Λ is the category of Artinian local Λ-algebras with residue field k.
- $\hat{\mathrm{Art}}_\Lambda$ is the category of complete noetherian local Λ-algebras $(R, \mathfrak{m}_R)$ such that for each $n \geq 1$ we have $R/\mathfrak{m}_R^n \in \mathrm{Art}_\Lambda$.

2.2 Deformations of a node

Definition 2.1. Let $\bar{A} := k[[x,y]]/(xy) \leftarrow k$. We will refer to $\bar{A}$ as the *standard node*. By a *deformation of the node (over R)* we will refer to tuples $(A \leftarrow R, \iota)$ where $R \in \hat{\mathrm{A}}\mathrm{rt}_\Lambda$, A is a complete local flat R-algebra and $\bar{A} \xleftarrow{\iota} A$ fits into a *Cartesian* diagram:

$$\begin{tikzcd} \bar{A} & A \arrow[l, "\iota"'] \\ k \arrow[u] & R \arrow[l] \arrow[u] \end{tikzcd}$$

where the map $k \leftarrow R$ is the residue map. Isomorphisms of deformations are defined in the usual way.

Definition 2.2. The *functor of deformations of the node* is the functor $G : \mathrm{Art}_\Lambda \to (\mathrm{Sets})$ which maps R to the set of isomorphism classes of deformations of the node over R.

The following theorem is folklore. The proof follows essentially the same steps as in [Stacks, Tag 0CBX].

Theorem 2.3. *The deformation $(\Lambda[[x,y,t]]/(xy-t) \leftarrow \Lambda[[t]], j : t \mapsto 0)$ is universal, i.e., $\Lambda[[\tau]]$ pro-represents G. In particular, given any deformation $(A \leftarrow R, \iota) \in G(R)$ we have a unique map $\Lambda[[t]] \to R : t \mapsto \pi \in \mathfrak{m}_R$ which induces an isomorphism $A \simeq R[[x,y]]/(xy-\pi)$.*

Remark 2.4. We can define a functor $\mathfrak{m} : \mathrm{Art}_\Lambda \to (\mathrm{Sets}) : R \mapsto \mathfrak{m}_R$ by attaching the maximal ideal to a local ring. Another way to interpret Theorem 2.3 is to say that G and $\mathfrak{m}$ are naturally isomorphic. More precisely, the natural transformation $G \to \mathfrak{m}$ can be defined as $(R[[x,y]]/(xy-\pi) \leftarrow R, \iota) \in G(R) \mapsto (\pi \in \mathfrak{m}_R)$.

2.3 Deformations of a root

Remark 2.5. Any line bundle on a curve restricted to the complete local ring of one of its nodes will be (non-canonically) isomorphic to the trivial line bundle. For this reason, we will study the roots of the trivial line bundle on a deformation of the node.

Set-up 2.6. Throughout this subsection let $(A \leftarrow R, \iota)$ be a deformation of the node and let E be an R-flat and R-relatively torsion-free rank-1 A-module, *which is not free.*

Remark 2.7. We exclude the case where E is free simply because its deformation theory is trivial. However, free roots play a role in later chapters.

To define the notion of a root we need to discuss bilinear forms momentarily.

Remark 2.8. With E as in Set-up 2.6, if $b : E^{\otimes 2} \to A$ is a bilinear form then b is symmetric. Indeed, since E is rank-1 the map $E^{\otimes 2} \twoheadrightarrow \mathrm{Sym}^2 E$ is generically an isomorphism, with the kernel being (x,y)-torsion. Since A has no (x,y)-torsion, b kills this kernel and factors through $\mathrm{Sym}^2 E$.

Definition 2.9. A tuple (E, b) with E as in Set-up 2.6 and with $b : \mathrm{Sym}^2 E \to A$ a non-degenerate bilinear form on E will be called a *root*. An isomorphism between two roots (E, b) and (E', b') is an isomorphism $\mu : E \to E'$ such that $b' \circ \mu^2 = b$. We will denote $b' \circ \mu^2$ by $\mu^* b'$.

Although we are excluding the case where E is free, we will often want to refer to this case. Hence we will also introduce the following terminology.

Definition 2.10. A tuple $(E, b : E^2 \xrightarrow{\sim} A)$ where E is a *free* rank-1 A-module will be referred to as a *free root*. We say (E, b) is a *possibly free root* if (E, b) is allowed to be either a free root or a (non-free) root.

Let $(\bar{E}, \bar{b})$ be a root on the standard node and (E, b) a root on $(A \leftarrow R, \iota)$. We will write $\iota_* E$ for $E \otimes_A \bar{A}$ and $\iota_* b$ for the map $\iota_* E^2 \to \bar{A}$ induced from b.

Definition 2.11. Let $j : \iota_* E \xrightarrow{\sim} \bar{E}$ be an isomorphism such that $\bar{b} \circ j^2 = \iota_* b$. Then we will refer to the tuple (E, b, j) as a *deformation of the root* $(\bar{E}, \bar{b})$. The map j will be called a *restriction map*. An isomorphism of deformations is an isomorphism of roots commuting with the restriction maps.

2.4 Standard roots

Let $R \in \hat{\mathrm{Art}}_\Lambda$ and $A = R[[x, y]]/(xy - \pi)$ for some $\pi \in \mathfrak{m}_R$. Define $\iota : A \to \bar{A} = k[[x, y]]/(xy)$ using $R \to R/\mathfrak{m}_R = k$.

2.4.1 Faltings' construction

Let $p, q \in R$ be such that $pq = \pi$. Define 2×2 matrices with entries in A:

$$\alpha = \begin{pmatrix} x & p \\ q & y \end{pmatrix}, \quad \beta = \begin{pmatrix} y & -p \\ -q & x \end{pmatrix}$$

Clearly $\alpha\beta = \beta\alpha = 0$ but moreover we get an exact infinite periodic complex (see [Fal96]):

$$\ldots \to A^{\oplus 2} \xrightarrow{\alpha} A^{\oplus 2} \xrightarrow{\beta} A^{\oplus 2} \xrightarrow{\alpha} A^{\oplus 2} \xrightarrow{\beta} A^{\oplus 2} \to \ldots$$

Definition 2.12. Define $E(p, q) \subset A^{\oplus 2}$ to be the image of α or, equivalently, the kernel of β. Truncating the complex above we get a free resolution of $E(p, q)$, whenever we refer to the standard resolution of $E(p, q)$ this is the one we mean.

Remark 2.13. It is straight forward to check that $E(p, q)$ is relatively torsion-free. Moreover, with some more work, one can see that E is R-flat, see Construction 3.2 of [Fal96].

If p or q is invertible, then $E(p, q)$ is free. As we are not dealing with free roots, from now on we assume $p, q \in \mathfrak{m}_R$. Note that this implies $\pi \in \mathfrak{m}_R^2$.

It is easy to see that the dual $E(p, q)^\vee = \hom(E(p, q), A)$ is naturally isomorphic to $E(q, p)$. In particular, when $p = q$ the module $E(p, p)$ is self-dual.

Definition 2.14. The natural pairing gives us a map $s : E(p,p)^2 \to A$ which we will call the *standard map*. For $\bar{E} = E(0,0)$ on $\bar{A}$ denote the standard map by $\bar{s}$.

Remark 2.15. Theorem 2.20 below states that any root on $(A \leftarrow R, \iota)$ is isomorphic to $(E(p,p), s)$ for some $p \in \mathfrak{m}_R$. In particular, we have a root iff $\pi \in \mathfrak{m}_R^2$.

Definition 2.16. Let us refer to $(E(p,p), s)$ as a *standard root on* $(A \leftarrow R, \iota)$.

Remark 2.17. There may be different values of p which give non-isomorphic roots. But on $\bar{A} \leftarrow k$ there is only one standard root. As an example take $R = k[t]/(t^2)$ and $A = R[[x,y]]/(xy)$. Then $E(t,t)$ and $E(0,0)$ are not isomorphic even as modules (one could apply Proposition 3.3 of [Fal96] to see this).

2.4.2 Properties of standard roots

Given any b on $E(p,q)$ we can lift it to $\operatorname{Sym}^2 A^{\oplus 2} \twoheadrightarrow \operatorname{Sym}^2 E(p,q)$ to get a morphism $\tilde{b} : \operatorname{Sym}^2 A^{\oplus 2} \to A$. Letting e_1, e_2 be the standard generators of $A^{\oplus 2}$ and e_1^2, e_1e_2, e_2^2 the corresponding generators of $\operatorname{Sym}^2 A^{\oplus 2}$ we may uniquely identify b with the values $b_0 := \tilde{b}(e_1^2), b_1 := \tilde{b}(e_1e_2), b_2 := \tilde{b}(e_2^2)$. By abuse of notation we will write $b = (b_0, b_1, b_2)$.

Lemma 2.18. *For a standard root $(E(p,p), s)$ we have $s = (x, p, y)$.*

Proof. Let $\langle \cdot, \cdot \rangle$ denote the natural pairing $A^{\oplus 2} \times A^{\oplus 2} \to A$. The identification of $E(p,p)^\vee$ with $E(p,p)$ makes it clear that if $e, f \in E(p,p)$ and $u, v \in A^{\oplus 2}$ are such that $e = \alpha(u)$ and $f = \alpha(v)$ then we have

$$s(e,f) = \langle u, \alpha(v) \rangle = \langle \alpha(u), v \rangle.$$

Now, direct computation yields the result. $\square$

Lemma 2.19. *Any root $(E(p,p), b)$ on $A \leftarrow R$ is isomorphic to $(E(p,p), s)$.*

Proof. Lemma 5.4.10 [Jar98] states that $b = (ax, b_1, awy)$ where $a \in A^*$ and $w \in R^*$ such that $wp = p$. Note here that as we are working with square roots of line bundles, the hypothesis of the cited lemma is satisfied (as stated in Corollary 5.4.9 *loc.cit.*).

Let v be a square root of w and consider the isomorphism $\mu : E \to E$ which descends from multiplication by $\left(\begin{smallmatrix} v & 0 \\ 0 & v^{-1} \end{smallmatrix}\right)$ on $A^{\oplus 2}$. Clearly $\mu^* b = a(vb_0, b_1, v^{-1}b_2) = va(x, v^{-1}b_1, y)$. By scaling E we may now assume $va = 1$ and $b = (x, b_1, y)$ where we changed b_1.

Since $\alpha(y,0) = \alpha(0,p)$ and $\alpha(0,x) = \alpha(p,0)$ we see that $pb_2 = yb_1$ and $pb_0 = xb_1$. Which means $y(b_1 - p) = x(b_1 - p) = 0$ (we used $wp = p$). But $\operatorname{Ann}_A(x,y) = 0$ hence $b_1 = p$. $\square$

Theorem 2.20 (Faltings)**.** *Let (E,b) be a root on A. Then $\exists p \in \mathfrak{m}_R$ such that $(E,b) \xrightarrow{\sim} (E(p,p), s)$.*

Proof. We are going to apply Theorem 3.7 in [Fal96] to torsion-free sheaves of rank-1. In fact, Faltings classifies non-degenerate quadratic forms on E whereas we have non-degenerate bilinear forms $b : E^2 \to A$ which is the same.

Faltings' Theorem implies that $(E,b) \simeq (E(p,p), b')$ for some $p \in \mathfrak{m}_R$ and b'. But now we can apply Lemma 2.19 to deduce the desired result. $\square$

We now wish to describe isomorphisms of roots. Since we know that all roots are isomorphic to $(E(p,p),s)$ for some $p \in \mathfrak{m}_R$ with $p^2 = \pi$, it suffices to calculate $\mathrm{Iso}((E(p,p),s),(E(q,q),s))$ for $p,q \in \mathfrak{m}_R$ such that $p^2 = q^2 = \pi$.

Let $e_1, e_2 \in A^{\oplus 2}$ be the *standard basis* and let $\xi_1, \xi_2 \in E(p,p)$ be the images of e_1 and e_2 respectively. Let us refer to ξ_1, ξ_2 as the *standard generators* of $E(p,p)$. Note that any automorphism of $E(p,p)$ can be lifted to a map $A^{\oplus 2} \to A^{\oplus 2}$.

Notation 2.21. If $\left(\begin{smallmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{smallmatrix}\right) : A^{\oplus 2} \to A^{\oplus 2}$ descends to $\mu \in \hom(E(p,p),E(q,q))$ then we will write $\mu = \left[\begin{smallmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{smallmatrix}\right]$.

Lemma 2.22. *We have:*

$$\mathrm{Iso}((E(p,p),s),(E(q,q),s)) = \{\left[\begin{smallmatrix} \varepsilon_1 & 0 \\ 0 & \varepsilon_2 \end{smallmatrix}\right] \mid \varepsilon_1,\, \varepsilon_2 \in \{\pm 1\},\, q = \varepsilon_1 \varepsilon_2 p\}$$

Proof. This is proven in a similar way to Proposition 4.1.12 of [Jar98], so we will give a sketch. An easy observation is that we can choose a lift of μ of the form

$$\begin{pmatrix} u_+(x) & v_+(x) \\ v_-(y) & u_-(y) \end{pmatrix}$$

where $u_+, v_+ \in R[[x]] \subset A$ and $u_-, v_- \in R[[y]] \subset A$. Now we simply have to calculate what it means to have $\mu^* s = s$ in terms of $u_\pm, v_\pm$. Using that x (resp. y) does not annihilate $R[[x]]$ (resp. $R[[y]]$) we see immediately that $v_\pm = 0$, $u_\pm \in \{\pm 1\}$ is forced. Then $q = u_+ u_- p$. $\square$

Definition 2.23. Let $\mu : E(p,q) \xrightarrow{\sim} E(p',q')$ be an isomorphism. Notice that the free resolutions attached to these modules canonically identify the central fibers with $k^{\oplus 2}$. Denote the restriction of μ to the central fibers by $\mu(0) : k^{\oplus 2} \to k^{\oplus 2}$.

Remark 2.24. Suppose $\mu : (E(p,p),s) \xrightarrow{\sim} (E(q,q),s)$. Then restricting μ^2 to the central fibers gives us $\mu^2(0) : (k^{\oplus 2})^2 \to (k^{\oplus 2})^2$. It is immediate to check that $\mu^2(0) = \mathrm{id}$ iff $\mu(0) = \pm\,\mathrm{id}$ and $\mu^2(0) \neq \mathrm{id}$ iff $\mu(0) = \pm\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. In other words, when $p \neq 0$ then $\mu^2(0) = \mathrm{id}$ iff $p = q$.

Definition 2.25. On $(A \leftarrow R, \iota)$ there is a *natural restriction map* from $E(p,q)$ to $\bar{E} = E(0,0)$ which is the map r completing the diagram below:

$$\begin{array}{ccc} A^{\oplus 2} & \xrightarrow{\iota} & \bar{A}^{\oplus 2} \\ \downarrow{\scriptstyle \alpha(p,q)} & & \downarrow{\scriptstyle \alpha(0,0)} \\ E(p,q) & \overset{r}{\dashrightarrow} & \bar{E} \end{array}$$

Remark 2.26. The triplet $(E(p,p),s,r)$ is a deformation of the root $(\bar{E},\bar{s})$.

Remark 2.27. Lemma 2.22 implies that choosing a restriction map r rigidifies the root. That is, $\mathrm{Aut}(E(p,p),s,r) = 1$. The following result takes this observation one step further.

Proposition 2.28. *Suppose that (E, b, j) is a deformation of $(\bar{E}, \bar{s})$. Then there exists precisely one $p \in R$ such that $(E, b, j) \simeq (E(p, p), s, r)$. Moreover, this isomorphism is unique.*

Proof. Uniqueness of the isomorphism follows from Remark 2.27. By Theorem 2.20 we know that $(E, b) \simeq (E(p, p), s)$ for some $p \in R$. Picking one such isomorphism we may assume $(E, b, j) = (E(p, p), s, j)$ for some j. However, with our choice of identification, j is not necessarily equal to the natural restriction r.

Let $\gamma = j \circ r^{-1} : (\bar{E}, \bar{s}) \to (\bar{E}, \bar{s})$. Then γ is uniquely defined by $\gamma(0) \in \{\pm \mathrm{id}, \pm\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)\}$. An isomorphism $\mu : (E(p, p), s) \xrightarrow{\sim} (E(q, q), s)$ commutes with j and r iff $\iota_* \mu : (\bar{E}, \bar{s}) \xrightarrow{\sim} (\bar{E}, \bar{s})$ is the inverse of γ. Having classified such μ in Lemma 2.22 we know that there exists precisely one q and one μ which will restrict to γ^{-1}. $\square$

2.5 Universal deformation

Definition 2.29. Call the tuple $(\bar{A} \leftarrow k, \bar{E}, \bar{s})$ the *rooted node*. A deformation of the node together with a deformation of the root, which looks like $(A \leftarrow R, i, E, b, j)$, will be called a *deformation of the rooted node*.

Let the functor $F : \mathrm{Art}_\Lambda \to (\mathrm{Sets})$ associate to R the isomorphism classes of deformations of the rooted node.

Theorem 2.30. *The ring $\Lambda[[\tau]]$ pro-represents F. The universal family is given by $(\Lambda[[x, y, \tau]](xy - \tau^2) \leftarrow \Lambda[[\tau]], \tau \mapsto 0, E(\tau, \tau), s, r)$.*

Proof. Given any deformation of the rooted node $(A \leftarrow R, \iota, E, b, j)$ we wish to show that there exists a unique map $\varphi : \Lambda[[\tau]] \to R$ such that A is canonically isomorphic to $\Lambda[[x, y, \tau]](xy - \tau^2) \otimes_{\Lambda[[\tau]]} R$ and $\varphi_*(E(\tau, \tau), s, r) \simeq (E, b, j)$. Furthermore, that this isomorphism is unique.

Proposition 2.28 shows that there exists a *unique* $p \in R$ such that (E, b, j) is (uniquely) isomorphic to $(E(p, p), s, r)$, moreover this implies $A = R[[x, y]]/(xy - \pi)$ with $\pi = p^2$. Define φ by $\tau \mapsto p$. Since the maps s and r are natural, the pushforward of $(E(\tau, \tau), s, r)$ is (uniquely) isomorphic to $(E(p, p), s, r)$.

Choosing any other map $\tau \mapsto q$ would give a root that is not isomorphic to (E, b, j). Thus we have proven the existence and uniqueness of the map φ of the desired form. $\square$

Remark 2.31. As in Remark 2.4, this theorem allows us to identify the functor F with the functor $\mathfrak{m} : R \mapsto \mathfrak{m}_R$. This time the identification is achieved by mapping $(R[[x, y]]/(xy - p^2), \iota, E(p, p), s, r) \in F(R)$ to $p \in \mathfrak{m}_R$. If we identify the functor of deformations of the node G with $\mathfrak{m}$ as in Remark 2.4 then the forgetful functor $F \to G$ corresponds to the squaring map $\mathfrak{m} \to \mathfrak{m} : (p \in \mathfrak{m}_R) \mapsto (p^2 \in \mathfrak{m}_R)$.

Chapter 3

Universal deformation of a node with multiple roots

Having fixed a positive integer m, we will suppress it from notation when referencing m-tuples of roots.

Definition 3.1. Let $(A \leftarrow R, \iota)$ be a deformation of the node. A *multiple root* is a tuple $(\mathcal{R}, \Phi)$ where $\mathcal{R} = (E_i, b_i)_{i=1}^m$ is a sequence of (non-trivial) roots and $\Phi = (h_i : E_1^2 \xrightarrow{\sim} E_i^2)$ is a sequence of isomorphisms with $h_1 := \mathrm{id}$ and the rest satisfying the following conditions:

- $\exists u_i : E_1 \xrightarrow{\sim} E_i$ such that $u_i^2 = h_i$
- $b_i \circ h_i = b_1$.

The data Φ will be called a *synchronization* on the sequence of roots $\mathcal{R}$. An *isomorphism of multiple roots* is a sequence of isomorphisms between the roots commuting with the synchronizations.

Remark 3.2. If E_1 is isomorphic to $E(0,0)$ then so is E_i for all i. Then, A must be the trivial deformation of the node over R. Unless this is the case, it follows from Remark 2.24 that the synchronization Φ is uniquely determined from the sequence of roots $\mathcal{R}$.

Remark 3.3. In other words, the definition above is symmetric and does not even require an ordering of the index set. Indeed, given a multiple root $(\mathcal{R}, \Phi)$ and any pair of indices $i, j \in \{1, \dots, m\}$ then we can define $h_{ij} := h_j \circ h_i^{-1} : E_i^2 \xrightarrow{\sim} E_j^2$ which will satisfy the two conditions above. Conversely, if we are given isomorphisms $(h_{ij})_{i,j}$ with $h_{ii} = \mathrm{id}$, h_{ij} the square of an isomorphism and $b_i = b_j \circ h_{ij}$ then $(h_i := h_{1i})$ gives us a synchronization.

Although we will stick to the definition above for the next two sections, we will eventually need to consider not just sequences of roots but sequences of possibly free roots. Hence we define the following:

Definition 3.4. Let $\mathcal{R}$ be a sequence of possibly free roots. Let Φ be a synchronization on the subsequence $\mathcal{R}'$ of $\mathcal{R}$ consisting of non-free roots. If $\mathcal{R}' \neq \emptyset$ then $(\mathcal{R}, \Phi)$ will be called a *generalized multiple root* and when $\mathcal{R}' = \emptyset$ then $\mathcal{R}$ itself will be called a *generalized multiple root.* As before Φ is called a *synchronization.*

3.1 Deformations of multiple roots

Let $(A \leftarrow R, \iota)$ be a deformation of the standard node $\bar{A} \leftarrow k$. Let $(\bar{\mathcal{R}}, \bar{\Phi})$ be a multiple root on the standard node.

Definition 3.5. A *deformation of* $(\bar{\mathcal{R}}, \bar{\Phi})$ on A is a tuple $(\mathcal{R}, \Phi, \mathfrak{j})$ where $(\mathcal{R}, \Phi)$ is a multiple root and $\mathfrak{j}$ is a sequence of restriction maps, i.e., a sequence of isomorphisms $\mathfrak{j} : \iota_* \mathcal{R} \xrightarrow{\sim} \bar{\mathcal{R}}$. Moreover, we ask that $\mathfrak{j}$ commute with the synchronization Φ and $\bar{\Phi}$.

3.1.1 Conventions

- On the standard node $\bar{A} \leftarrow k$ denote the i-th root by $(\bar{E}_i, \bar{s}_i)$ and let each of these be the standard root, i.e., $(\bar{E}_i, \bar{s}_i) = (\bar{E}, \bar{s})$. As described in Remark 2.24, we have two options for each of the maps $\bar{h}_i : \bar{E}_1^2 \to \bar{E}_i^2$. Either $h_i = \mathrm{id}$ or $h_i \neq \mathrm{id}$. Fix a sequence $\bar{\Phi} = (\bar{h}_i)_{i=1}^m$. From now on $(\bar{\mathcal{R}}, \bar{\Phi})$ will denote this multiple root.

- Define a sequence $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m)$ where $\varepsilon_1 = 1$, $\varepsilon_i = 1$ if $\bar{h}_i = \mathrm{id}$ and $\varepsilon_i = -1$ if $\bar{h}_i \neq \mathrm{id}$. Clearly we can recover $\bar{\Phi}$ from ε.

- Given a deformation $(\mathcal{R}, \Phi, \mathfrak{j})$ use the following letters for the underlying objects: $\mathcal{R} = (E_i, b_i)_{i=1}^m$, $\Phi = (h_i)_{i=1}^m$ and $\mathfrak{j} = (j_i)_{i=1}^m$.

- In the rest of this section, equality between deformations of roots is used to designate the unique isomorphism between them.

3.1.2 The universal deformation

Lemma 3.6. *Let $(\mathcal{R}, \Phi, \mathfrak{j})$ be a deformation of $(\bar{\mathcal{R}}, \bar{\Phi})$ on $(A \leftarrow R, \iota)$. Then $\exists! p \in R$ such that for all i we have $(E_i, b_i, j_i) = (E(\varepsilon_i p, \varepsilon_i p), s, r)$.*

Proof. By Proposition 2.28 we know that $\forall i\ \exists! p_i \in R$ such that $(E_i, b_i, j_i) = (E(p_i, p_i), s, r)$. Let $p = p_1$. Since the existence of Φ forces all roots (E_i, b_i) to be isomorphic we may apply Lemma 2.22 to conclude $p_i \in \{\pm p\}$.

If $p = 0$ then there is nothing more to prove so assume $p \neq 0$. Then the sign of p_i is completely determined by $h(0)$ by Remark 2.24. However, the restriction maps r identify all the central fibers of E_i and $\bar{E}_i$ so that $h(0) = \mathrm{id}$ iff $\bar{h}(0) = \mathrm{id}$. Thus $p_i = p$ iff $\varepsilon_i = 1$. □

Define $H : \mathrm{Art}_\Lambda \to (\mathrm{Sets})$ to be the functor associating to each R the set of isomorphism classes of deformations of $(\bar{\mathcal{R}}, \bar{\Phi})$.

Theorem 3.7. *The functor H is pro-represented by $\Lambda[[\tau]]$ with the universal deformation given by $(\Lambda[[\tau, x, y]]/(xy - \tau^2) \leftarrow \Lambda[[\tau]], \tau \mapsto 0, (E(\varepsilon_i \tau, \varepsilon_i \tau), s, r)_{i=1}^m)$.*

Remark 3.8. We omitted the synchronizations from the description of the family because by Remark 3.2 the synchronizations are uniquely defined given these roots.

Proof. Let $(A \leftarrow R, \iota)$ be a deformation of the node and let $(\mathcal{R}, \mathfrak{j}) = (E_i, b_i, j_i)_{i=1}^m$ together with the synchronizations $\Phi = \{h_i\}_{i=1}^m$ be a deformation of $(\bar{\mathcal{R}}, \bar{\Phi})$. Any map $\varphi : \Lambda[[\tau]] \to R$ is uniquely defined by the choice of $p \in R$ for which $\tau \mapsto p$. Lemma 3.6 tells us that there is a unique $p \in R$ for which $\varphi_*(E(\varepsilon_i\tau, \varepsilon_i\tau), s, r) = (E_i, b_i, j_i)$. This proves the existence and uniqueness of φ provided we show that the synchronizations agree.

This is done by reducing to the fiber over the node as in Lemma 3.6 at which point compatibility of the synchronizations is immediate. ☐

We will be interested in generalized multiple roots and their deformations. Here is a key lemma which implies that the deformation functor of a multiple root does not change if we add free roots. See Remark 3.11 for a precise statement.

This time let $R \in \hat{\mathrm{A}}\mathrm{rt}_\Lambda$ and $(A \leftarrow R, \iota)$ be a deformation of the standard node. Note that A is complete with respect to the ideal $\mathfrak{m}_R \cdot A$. For the following lemma, we can allow $\bar{A}$ to be a smooth k-algebra.

Lemma 3.9. *Let $(\bar{L}, \bar{b} : \bar{L}^{\otimes 2} \xrightarrow{\sim} \bar{A})$ be a free root on $\bar{A} \leftarrow k$ and let $(A \leftarrow R)$ be a deformation of $\bar{A} \leftarrow k$. Then there exists a unique deformation of $(\bar{L}, \bar{b})$ on $(A/R, \iota)$, up to unique isomorphisms.*

Proof. Since A is complete with respect to $\mathfrak{m}_R \cdot A$ we just have to show that there exists a unique lift of the root from $R/\mathfrak{m}_R^n$ to $R/\mathfrak{m}_R^{n+1}$. Existence is clear. What has to be shown is that there exists a unique isomorphisms between any two lifts. But this can be reduced to showing that successive lifts of square roots of invertible elements are unique, which is true. ☐

Remark 3.10. The proof of this lemma works in greater generality, and we need the more general statement as well. If we have a deformation of a nodal curve then in Zariski neighbourhoods of smooth points, the deformation is trivial. In particular, with the base complete local, the argument above works verbatim. Since roots are always free on the smooth locus of the fibers, we can lift any root uniquely along the smooth part of a deformation.

Remark 3.11. Lemma 3.9 has the following useful consequence. Let $\bar{\mathcal{R}}$ be a generalized root and $\bar{\mathcal{R}}'$ the multiple root obtained from $\bar{\mathcal{R}}$ by removing the free roots. Then, Lemma 3.9 implies that the deformation functors of $\bar{\mathcal{R}}$ and $\bar{\mathcal{R}}'$ are identified by forgetting the free roots. Here, if $\bar{\mathcal{R}}' = \emptyset$ then by a deformation of $\bar{\mathcal{R}}'$ we will mean the deformation of the underlying node.

3.2 Further comments on our definition of multiple roots

Instead of working with a sequence of roots, we chose to work with

(a) a sequence of *isomorphic* roots,

(b) together with the square of an isomorphism between each pair of roots.

Suppose we dropped these 2 conditions and defined the functor F_m of m-tuples of roots without any restrictions. Then clearly F_m is the m-fold product of F (deformation of roots) over G (deformation of nodes). As both of these functors are pro-representable, so is F_m. For instance $F_2 = F \times_G F$ is pro-represented by $\Lambda[[\tau]]] \times_{\Lambda[[\tau^2]]} \Lambda[[\tau]] \simeq \Lambda[[\tau_1, \tau_2]]/(\tau_1^2 - \tau_2^2)$, which is non-normal.

What if we include the first condition and drop the second condition? Define the functor F'_m of m-tuples of *isomorphic* roots.

Claim 3.12. *The functor F'_m is not pro-representable.*

Proof. For convenience let $m = 2$ and $F' := F'_2$, though the proof works for all $m > 1$. Recalling that the functor of deformation of roots, F, is identified with $R \mapsto \mathfrak{m}_R$ it becomes clear that $F' : R \mapsto \{(p, \varepsilon p) \mid p \in \mathfrak{m}_R, \varepsilon = \pm 1\}$.

To show that the functor F' is not pro-representable we check Schlessinger's first condition (denoted by H_1 in [Sch68]). Let $k[\epsilon_i] \simeq k[t]/(t^2)$ and $k[\epsilon_1, \epsilon_2] = k[\epsilon_1] \times_k k[\epsilon_2]$. Then the map $F'(k[\epsilon_1, \epsilon_2]) \to F'(k[\epsilon_1]) \times F'(k[\epsilon_2])$ is given by $(a\epsilon_1 + b\epsilon_2, \nu) \mapsto (a\epsilon_1, \nu) \times (b\epsilon_2, \nu)$ where $\nu \in \{\pm 1\}$. Since the pair of signs has to be equal on the image (both equal to ν), this map can not be surjective. Thus H_1 is violated and F' can not be pro-representable. □

On the other hand, if one were to pick a sequence of roots together with isomorphisms between them (as opposed to the square of the isomorphisms) then one simply recovers the functor F_1. This defeats the purpose, because pairs of free roots are locally isomorphic but since we only identify the squares of these line bundles, we can only determine a local isomorphism up to sign. Thus, F_1 does not adequately generalize the moduli problem of having m locally free roots on smooth curves.

Chapter 4

Universal deformation of stable curves with multiple roots

4.1 Universal deformation of a stable curve

Let X/k be a stable curve of genus g with n nodes $x_1, \ldots, x_n \in X$.

Set-up 4.1. For each i, let $\hat{\mathcal{O}}_i := \hat{\mathcal{O}}_{X,x_i} \simeq k[[x,y]]/(xy)$. Let $G_i : \mathrm{Art}_\Lambda \to (\mathrm{Sets})$ be the functor of deformations of $\hat{\mathcal{O}}_i \leftarrow k$. By Theorem 2.3 we can pick a formal variable t_i and identify G_i with $\hom(\Lambda[[t_i]], \_)$.

Definition 4.2. For $R \in \hat{\mathrm{Art}}_\Lambda$, a curve $\mathcal{X}/R$ together with a k-isomorphism $\iota : X \xrightarrow{\sim} \mathcal{X}|_k$ is said to be a *deformation of X over R*. Pullbacks and isomorphisms of deformations are defined in the usual way.

Remark 4.3. When R is artinian, the underlying topological spaces of a deformation $\mathcal{X}/R$ of X and X itself are naturally identified. We will use this identification between the points of X and points of $\mathcal{X}$ without further remark.

Definition 4.4. The functor $D_X : \mathrm{Art}_\Lambda \to (\mathrm{Sets})$ assigning to each R the set of isomorphism classes of deformations of X over R is called the *functor of infinitesimal deformations of X*.

Given a deformation $(\mathcal{X}/R, \iota)$ of X, the tuple $(\hat{\mathcal{O}}_{\mathcal{X},x_i} \leftarrow R, \iota)$ is a deformation of $\hat{\mathcal{O}}_i \leftarrow k$. Thus, for each of the nodes $x_i \in X$ we get a natural transformation $D_X \to G_i$.

Theorem 4.5 (Deligne-Mumford [DM69]). *Let $T = \Lambda[[t_1, \ldots, t_{3g-3}]]$. There exists a universal deformation $(\mathcal{C}/T, u)$ of X over T, through which T pro-represents D_X. Moreover, for each $i = 1, \ldots, n$, the map $D_X \to G_i$ corresponds to the map $\Lambda[[t_i]] \to T : t_i \mapsto t_i$.*

4.1.1 Deforming the line bundle

We need to pause for a minute and consider our situation before we can proceed further. Although we studied the universal deformation of the curve X as well as the universal deformation of each of its nodes, with and without roots, we can not make direct use of them. This is because an arbitrary line bundle N on X will, in general, not extend to the universal deformation of X.

We deal with this problem by specifying a global framework in which we specify deformations of the tuple (X, N) before we consider deformations of the roots of N. This is the reason for our set-up in Section 1.2, which we recall again.

We have a genus g stable curve $\mathcal{C} \to \mathcal{M}$ over a Deligne–Mumford stack $\mathcal{M}$ of finite type over an excellent scheme $S \to \mathbb{Z}[\frac{1}{2}]$. In addition, we specified a line bundle $\mathcal{N}$ on $\mathcal{C}$. Now, by a curve X/k together with a line bundle $\bar{\mathcal{N}}$ we really mean a morphism $\operatorname{Spec} k \to \mathcal{M}$. Pulling back $\mathcal{C} \to \mathcal{M}$ and $\mathcal{N}$ to $\operatorname{Spec} k$ gives us $(X/k, \bar{\mathcal{N}})$.

Note however that the pair $(X, \bar{\mathcal{N}})$ does not determine the map $\operatorname{Spec} k \to \mathcal{M}$, since $\mathcal{M}$ need not be universal in anyway. So we really need to specify the morphism $\operatorname{Spec} k \to \mathcal{M}$ and not just the tuple $(X, \bar{\mathcal{N}})$. Nevertheless, for readability, we may often refer to a morphism $\operatorname{Spec} k \to \mathcal{M}$ simply as $(X, \bar{\mathcal{N}})$. Here is one example of how we will abuse notation.

Definition 4.6. Let k be any field and R a complete local ring with residue field k. Let $(X, \bar{\mathcal{N}})$ be obtained from $p : \operatorname{Spec} k \to \mathcal{M}$. By a *deformation of* $(X, \bar{\mathcal{N}})$ *on* R (but really a deformation of p on R) we will mean a tuple (P, ι) such that the following diagram 2-commutes:

$$\begin{tikzcd} \operatorname{Spec} k \arrow[d] \arrow[dr, bend left, "p"] \arrow[dr, phantom, "\iota \Downarrow"] & \\ \operatorname{Spec} R \arrow[r, "P"'] & \mathcal{M} \end{tikzcd}$$

where the vertical arrow is the quotient map $R \twoheadrightarrow k$.

Remark 4.7. If P gives us $(\mathcal{X}/R, \mathcal{N})$ then ι is an isomorphism $(\mathcal{X}|_k, \mathcal{N}|_k) \xrightarrow{\sim} (X, \bar{\mathcal{N}})$ over k. For this reason, in accordance with our previous notation, we will denote deformations of $(X, \bar{\mathcal{N}})$ as tuples $(\mathcal{X}/R, \mathcal{N}, \iota)$.

4.1.2 Geometric formal neighbourhoods

Definition 4.8. Let k be any field. Then by Cohen structure theorem there exists a universal coefficient ring, which we will denote by $\mathfrak{o}_k$. So that any complete local ring with residue field k contains a copy of $\mathfrak{o}_k$. If $\operatorname{char} k = 0$ then $\mathfrak{o}_k = k$.

Let $s : \operatorname{Spec} k \to S$ be any point. The complete local ring $\hat{\mathcal{O}}_{S,s}$ pro-represents the functor $\mathrm{Art}_{\mathfrak{o}_k} \to (\mathrm{Sets})$ defined by $A \mapsto \hom_s(A, S)$, where the subscript s indicates that the morphisms must restrict to s on the residue field.

For any morphism $s : \operatorname{Spec} k \to S$, with k any field we can still define a functor $Q_s : \mathrm{Art}_{\mathfrak{o}_k} \to (\mathrm{Sets})$ via the rule $A \mapsto \hom_s(A, S)$. If $s \to S$ factors through the point $s' : \operatorname{Spec} k' \to S$ then Q_s is pro-represented by the complete local ring $\hat{\mathcal{O}}_{S,s'} \otimes_{\mathfrak{o}_{k'}} \mathfrak{o}_k$.

Definition 4.9. For any point $s : \operatorname{Spec} k \to S$ let $\hat{\mathcal{O}}_{S,s}$ denote the complete local ring pro-representing the functor Q_s above.

For the Deligne–Mumford stack $\mathcal{M}$ and a point $p : \operatorname{Spec} k \to \mathcal{M}$ the functor Q_p can be defined just like Q_s. This functor is seen to be pro-representable by using any étale chart.

Definition 4.10. For any point $p : \operatorname{Spec} k \to \mathcal{M}$ the complete local ring pro-representing Q_p will be denoted by $\hat{\mathcal{O}}_{\mathcal{M},p}$.

Let $s : \operatorname{Spec} k \to S$ be a geometric point and $p : \operatorname{Spec} k \to \mathcal{M}$ be a point of finite type lying above s. Let $\Lambda = \hat{\mathcal{O}}_{S,s}$ and let $\tilde{\Lambda} := \hat{\mathcal{O}}_{\mathcal{M},p}$. Denote by X/k the fiber of $\mathcal{C} \to \mathcal{M}$ over p and let $\bar{\mathcal{N}}$ be the restriction of $\mathcal{N}$ to X.

By Theorem 4.5 the universal deformation functor D_X of X can be represented by the ring $\Lambda[[t_1, \dots, t_{3g-3}]]$. For convenience let us refer to $\hom_{\hat{\mathcal{O}}_{S,s}}(\hat{\mathcal{O}}_{\mathcal{M},p}, \_)$ as $D_{\mathcal{M},p}$. The natural map $D_{\mathcal{M},p} \to D_X$ corresponds to a map:

$$\Lambda[[t_1, \dots, t_{3g-3}]] \to \tilde{\Lambda}.$$

We will study how introducing multiple roots changes this local behavior. To do this, let us first define deformations over $\mathcal{M}$.

4.2 Multiple roots on nodal curves

We will begin with a single root on a nodal curve. Let $\bar{\mathcal{N}}$ be a line bundle on the stable curve X/k. Let $(\mathcal{X}/R, \mathcal{N}, \iota)$ be a deformation of $(X, \bar{\mathcal{N}})$.

Definition 4.11. For each node $x_v \in X$ denote by $U_{x_v}(\mathcal{X}) := \operatorname{Spec} \hat{\mathcal{O}}_{\mathcal{X},x_v} \to \mathcal{X}$ the *formal neighbourhood of* x_v *in* $\mathcal{X}$. The pullback map to $U_{x_v}(\mathcal{X})$ will be denoted by $\hat{x}_v^*$.

Remark 4.12. Given a root $(\mathcal{E}, b)$ of $\mathcal{N}$ on $\mathcal{X}$, the pullback $\hat{x}_v^*(\mathcal{E}, b)$ is a (possibly free) root on $U_v(\mathcal{X})$, in the sense of Definition 2.10 (keeping in mind Remark 2.5). The codomain of $\hat{x}_v^* b$ is no longer canonically isomorphic to the structure sheaf but to $\hat{x}_v^*\mathcal{N}$. This makes no difference for the theory in the sense that the deformation functors are isomorphic.

As we move on to sequences of roots, let us recall our fixed integer $m \geq 1$. Suppose we have a sequence $\mathcal{R} := (\mathcal{E}_i, b_i)_{i=1}^m$ of roots on $\mathcal{X}$. Then $\hat{x}_v^*\mathcal{R}$ is a sequence of (possibly free) roots. We want to define a synchronization for such sequences.

Definition 4.13. For a root $(\mathcal{E}, b)$ on $\mathcal{X}$, by *singularities of* $\mathcal{E}$ we will mean the subset of the nodes on which $\mathcal{E}$ is not-free.

Re-index the nodes if necessary so that we get an $n' \leq n$ such that x_v is a singularity of one of the roots if and only if $v \leq n'$.

Definition 4.14. Let $\mathcal{R}_{x_v} := \hat{x}_v^*\mathcal{R}$ be the sequence of possibly free roots on $U_v(\mathcal{X})$. A synchronization, as in Definition 3.4, on $\mathcal{R}_{x_v}$ will be denoted by Φ_{x_v}.

Definition 4.15. Let $\mathcal{R}$ be a sequence of roots on $\mathcal{X}$. For each $v \leq n'$ let Φ_{x_v} be a synchronization on $\mathcal{R}_{x_v}$. Let $\Phi = (\Phi_{x_v})_{v=1}^{n'}$. Then the pair $(\mathcal{R}, \Phi)$ will be called a *multiple root (of $\mathcal{N}$).*

Remark 4.16. If for some v the sequence $\mathcal{R}_{x_v}$ contains non-isomorphic roots, then there exists no synchronization Φ_{x_v}. In this case, $\mathcal{R}$ simply can not be made into a multiple root.

4.3 Deformation theory

Fix a multiple root $(\bar{\mathcal{R}}, \bar{\Phi})$ on X/k. Let $\xi = (X/k, \bar{\mathcal{R}}, \bar{\Phi})$.

Definition 4.17. If $(\mathcal{R}, \Phi)$ is a root on the deformation $(\mathcal{X}/R, \iota)$ and $\mathfrak{j} : \iota^*\mathcal{R} \xrightarrow{\sim} \bar{\mathcal{R}}$ is a sequence of isomorphisms then we will call $\mathfrak{j}$ a *restriction map.* If $\mathfrak{j}$ commutes with the synchronization Φ and $\bar{\Phi}$ then $(\mathcal{R}, \Phi, \mathfrak{j})$ will be called a *deformation of* $(\bar{\mathcal{R}}, \bar{\Phi})$, and $(\mathcal{X}/R, \iota, \mathcal{R}, \Phi, \mathfrak{j})$ is called a *deformation of* $(X/k, \bar{\mathcal{R}}, \bar{\Phi})$

Set-up 4.18. As before, order the nodes so that there is an $n' \leq n$ such that $\bar{\mathcal{R}}$ has non-free roots at x_v iff $v \leq n'$. Then, for any deformation $(\mathcal{R}, \Phi, \mathfrak{j})$ of $(\bar{\mathcal{R}}, \bar{\Phi})$ one of the roots in $\hat{x}_v^*\mathcal{R}$ is non-free iff $v \leq n'$.

Definition 4.19. The functor $D_\xi : \mathrm{Art}_\Lambda \to (\mathrm{Sets})$ which takes R to the set of isomorphism classes of deformations of ξ is called the *functor of infinitesimal deformations of* ξ.

Let $(\mathcal{X}/R, \iota)$ be a deformation of X/k where we allow $R \in \hat{\mathrm{Art}}_\Lambda$. Denote $\bar{\mathcal{R}}$ as $(\bar{\mathcal{E}}_i, \bar{b}_i)_{i=1}^m$.

Lemma 4.20. *Suppose that for each $1 \leq v \leq n'$ we are given a deformation $\mathfrak{R}_v = (\mathcal{R}_v, \Phi_v, \mathfrak{j}_v)$ of $\hat{x}_v^*(\bar{\mathcal{R}}, \bar{\Phi})$ on $U_v(\mathcal{X})$. Then there exists a unique deformation $(\mathcal{R}, \Phi, \mathfrak{j})$ of $\bar{\mathfrak{R}} = (\bar{\mathcal{R}}, \bar{\Phi})$ on $\mathcal{X}$ which pulls back to $(\mathcal{R}_v, \Phi_v, \mathfrak{j}_v)$ on each $U_v(\mathcal{X})$.*

Proof. We use the Grothendieck Existence Theorem to reduce the question to the formal neighbourhood of the central fiber. Thus we may assume R is artinian.

Let $\mathfrak{m}_R$ be the maximal ideal of R. Let $R_l := R/\mathfrak{m}_R^l$ and $X_l := \mathcal{X}|_{R_l}$ for all $l \geq 0$. For each l and $v \leq n'$ we can pullback $\mathfrak{R}_v$ to the formal neighbourhood of the v-th node on X_l. We will denote this local deformation by $\mathfrak{R}_{v,l}$.

Using induction, we fix $N \geq 0$ and suppose that there is a unique deformation of the multiple root $\bar{\mathfrak{R}}$ on X_N such that for all $v \leq n'$ this deformation agrees with $\mathfrak{R}_{v,N}$ around the node x_v.

Constructing a lift of this deformation to X_{n+1} and showing that this lift is unique up to unique isomorphism will end the proof. We will do this by fpqc-descent on X_{n+1}. The synchronized roots around the formal neighbourhoods of the nodes are one portion of the descent data. For the rest of the descent data, we will construct the root away from the nodes and then show compatibility.

On the complement W of the nodes $x_1, \ldots, x_{n'}$, the roots we have are all free. Use Lemma 3.9 and Remark 3.10 to conclude that each root deforms uniquely in W.

This uniqueness also proves compatibility with the formal neighbourhoods around the nodes. $\square$

In Set-up 4.1 we denoted by G_v the deformation functor of the node $x_v \in X$, i.e. of the algebra $\hat{\mathcal{O}}_{X,x_v} \leftarrow k$.

Since we need to work with deformations over $\mathcal{M}$, even the nodes will not freely deform. The possible deformations of the node x_i is captured by the image of the map $D_{\mathcal{M},p} \to G_i$.

Let $\mathcal{Y} \to \operatorname{Spec} \tilde{\Lambda}$ be the universal deformation of X over $\mathcal{M}$. In other words, $\tilde{\Lambda} = \hat{\mathcal{O}}_{\mathcal{M},p}$ and $\mathcal{Y}$ is the pullback of $\mathcal{C} \to \mathcal{M}$. Let $U_i = \operatorname{Spec} \hat{\mathcal{O}}_{\mathcal{Y},x_i}$ be the formal neighbourhood of $x_i \in \mathcal{Y}$. Choose a trivialization of $\mathcal{N}$ on U_i.

If $(\mathcal{X}/R, \iota) \in D_{\mathcal{M},p}$ is any deformation of X/k over $\mathcal{M}$, then the formal neighbourhood $\operatorname{Spec} \hat{\mathcal{O}}_{\mathcal{X},x_i}$ will factor through U_i, hence the trivialization of $\mathcal{N}$ on U_i pulls back to $\operatorname{Spec} \hat{\mathcal{O}}_{\mathcal{X},x_i}$. In this way, we can identify deformations of roots of the trivial bundle with the deformations of roots of $\mathcal{N}$ (around the nodes).

Let F_v denote the deformation functor of the node x_v and the (generalized) multiple root $\hat{x}_v^*(\bar{\mathcal{R}}, \bar{\Phi})$, now viewed as a multiple root of the trivial bundle via our identification. This last point is crucial in actually defining F_v, because the line bundle $\mathcal{N}$ can not be made sense of for a general deformation of the node.

Finally, we are ready to state the main technical result we have been building up to since the last three sections.

Theorem 4.21. *The map $D_\xi \to D_{\mathcal{M},p} \times_{G_1} F_1 \times_{G_2} F_2 \cdots \times_{G_{n'}} F_{n'}$ is an isomorphism.*

Proof. We described the map above, but let us summarize it here. The map $D_\xi \to D_{\mathcal{M},p}$ just forgets the roots. Now, passing to the formal neighbourhood of the i-th node gives $D_{\mathcal{M},p} \to G_i$.

Pulling back the tuple of roots to the formal neighbourhood of x_i gives us a map $D_\xi \to F_i$. To define this map properly, we used trivializations of $\mathcal{N}$ around the universal deformation of X over $\mathcal{M}$ hence the maps $D_\xi \to F_i$ are not canonical. Forgetting the root here gives us a map to G_i.

The main technical difficulty in establishing this result is the construction of the inverse map. This inverse is given in Lemma 4.20. $\square$

Since all of these functors are pro-representable we conclude that D_ξ is also pro-representable. In fact, representing G_i with $\Lambda[[t_i]]$ and F_i with $\Lambda[\tau_i]/(\tau_i^2 - t_i)$ we calculate that D_ξ is represented by the ring T' below:

$$T' := \left(\left(\cdots \left(\tilde{\Lambda} \otimes_{\Lambda[[t_1]]} \Lambda[\tau_1]/(\tau_1^2 - t_1)\right) \otimes_{\Lambda[[t_2]]} \cdots\right) \otimes_{\Lambda[[t_{n'}]]} \Lambda[\tau_{n'}]/(\tau_{n'}^2 - t_{n'})\right).$$

Denoting the image of t_i in $\tilde{\Lambda}$ by $\tilde{t}_i$ we can write this as follows:

$$T' = \tilde{\Lambda}[\tau_1, \ldots, \tau_{n'}]/(\tau_1^2 - \tilde{t}_1, \ldots, \tau_{n'}^2 - \tilde{t}_{n'}).$$

Recall that $\tilde{t}_i$ describes how the node x_i deforms in $\mathcal{Y} \to \operatorname{Spec} \tilde{\Lambda}$ in the sense that $\hat{\mathcal{O}}_{\mathcal{Y},x_i} \simeq \Lambda[[x,y]]/(xy - \tilde{t}_i)$.

Furthermore, Lemma 4.20 implies that D_ξ is not just pro-represented by T' but this representation is *effective*: there is a universal deformation of ξ over T'.

Recalling Set-up 4.18 regarding our convention for n' we summarize all this in the following statement.

Corollary 4.22. *Let $p : \operatorname{Spec} k \to \mathcal{M}$ give a pair $(X, \bar{\mathcal{N}})$ and let $\xi = (X, \bar{\mathcal{N}}, \bar{\mathcal{R}}, \bar{\Phi})$. The local deformation functor D_ξ of ξ over $\mathcal{M}$ is pro-represented by*

$$T' = \hat{\mathcal{O}}_{\mathcal{M},p}[\tau_1, \ldots, \tau_{n'}]/(\tau_1^2 - \tilde{t}_1, \ldots, \tau_{n'}^2 - \tilde{t}_{n'})$$

such that the forgetful map $D_\xi \to D_{\mathcal{M},p}$ corresponds to the inclusion $\hat{\mathcal{O}}_{\mathcal{M},p} \hookrightarrow T'$. Moreover, there is a universal deformation of ξ over T' making this representation effective.

Perhaps one of the most important application of this result is to $\mathcal{M} = \overline{\mathcal{M}}_g$ with the universal curve over it. Then, $\mathcal{N}$ is $\omega_{\mathcal{C}/\mathcal{M}}^{\otimes l}$ for some $l \in \mathbb{Z}$.

In this case $\mathcal{O}_{\mathcal{M},p} \simeq T = \Lambda[[t_1, \ldots, t_{3g-3}]]$. Let $T' = \Lambda[[\tau_1, \ldots, \tau_{3g-3}]]$ so that the map $T \to T'$ below corresponds to the forgetful functor $D_\xi \to D_X$:

$$T \to T' : t_i \mapsto \begin{cases} \tau_i^2 & : \quad i \le n' \\ \tau_i & : \quad i > n' \end{cases} . \tag{4.3.1}$$

These explicit calculations of the representing ring allows us to conclude the following result. We state it in this following weak format since we don't have the appropriate moduli space of multiple roots yet. The complete version is stated in Theorem 5.48.

Corollary 4.23. *If the natural map $\mathcal{M} \to \overline{\mathcal{M}}_g$ is smooth, then $D_\xi \to \hom(\hat{\mathcal{O}}_{S,s}, \_)$ is smooth.*

Proof. Recall $s \in S$ is the image of $m \in \mathcal{M}$ corresponding to X. So we need only observe that D_ξ is represented by the following ring:

$$\hat{\mathcal{O}}_{\mathcal{M},p} \otimes_{\Lambda[[t_1,\ldots,t_{3g-3}]]} \Lambda[[\tau_1, \ldots, \tau_{3g-3}]]$$

where $\Lambda[[t_1, \ldots, t_{3g-3}]] \to \hat{\mathcal{O}}_{\mathcal{M},p}$ is the natural map and the other map is defined in Equation 4.3.1 above. Note that we obtain this isomorphism not through some moduli interpretation but rather by calculation.

Recalling that $\Lambda = \hat{\mathcal{O}}_{S,s}$ the result follows from the observation that smoothness is preserved under pullback and composition. $\square$

4.4 Inessential automorphisms of multiple roots

Let X/k be a stable curve and N a line bundle on X. For $i = 1, \ldots, m$ let $\overline{\mathcal{R}}_i = (\overline{\mathcal{E}}_i, \alpha_i : \overline{\mathcal{E}}_i^2 \to N)$ be distinct roots of N. We will denote by $\xi_i = (X, \overline{\mathcal{R}}_i)$ the rooted curve. Let $\overline{\mathcal{R}}_0 = (\overline{\mathcal{R}}_1, \ldots, \overline{\mathcal{R}}_m)$ and $\overline{\Phi}$ be a synchronization of $\overline{\mathcal{R}}_0$. Let ξ_0 stand for $(X, \overline{\mathcal{R}}_0, \overline{\Phi})$.

For each $i \ge 0$ let $\operatorname{Aut}(\xi_i)$ stand for the group of automorphisms of ξ_i. Elements of this automorphism group are tuples (f, φ) where $f : X \xrightarrow{\sim} X$ and $\varphi : \overline{\mathcal{R}}_i \xrightarrow{\sim} \varphi^* \overline{\mathcal{R}}_i$, with the condition that if $i = 0$ then φ respects the synchronization $\overline{\Phi}$.

Definition 4.24. By $\mathrm{Aut}_0(\xi_i) \subset \mathrm{Aut}(\xi_i)$ we will denote the subgroup consisting of automorphisms (f, φ) where $f = \mathrm{id}_X$. Following the convention of [Cor89], an automorphism in $\mathrm{Aut}_0(\xi_i)$ will be called an *inessential automorphism* (see Remark 4.29).

Notation 4.25. By $\underline{\mathrm{Aut}}_0$ and $\underline{\mathrm{Aut}}$ we will mean $\mathrm{Aut}_0/\{\pm 1\}$ and $\mathrm{Aut}/\{\pm 1\}$ respectively.

For $i = 1, \dots, m$ let $W_i \subset X$ be the set of nodes for which $\overline{\mathcal{R}}_i$ is singular and let $W_0 = \cup_{i=1}^m W_i$. Let $\nu_i : X_i \to X$ be the partial normalization of X at W_i. We will denote by V_i the set of connected components of X_i.

Following [Cor89] and [CCC07] we define the following graphs. View $\Gamma_i = (V_i, W_i)$ as a graph with vertex set V_i and edge set W_i, and to an "edge" $x \in W_i$ we associate the "end points" in V_i which are the components containing a preimage of x.

Using the coefficient ring $\{\pm 1\}$, and suppressing it from notation, we construct the cohomology chain complexes to each graph:

$$0 \longrightarrow \mathrm{H}^0(\Gamma_i) \longrightarrow C^0(\Gamma_i) \xrightarrow{\partial} C^1(\Gamma_i) \longrightarrow \mathrm{H}^1(\Gamma_i) \longrightarrow 0$$

Note that, by virtue of our coefficient ring, we do not need an orientation on Γ_i for the coboundary map to be well defined.

Lemma 4.26 ([Cor89],[CCC07])**.** *For $i > 0$ we have $\mathrm{Aut}_0(\xi_i) \simeq C^0(\Gamma_i)$ and $\underline{\mathrm{Aut}}_0(\xi_i) \simeq \ker(C^1(\Gamma_i) \to \mathrm{H}^1(\Gamma_i))$.*

Proof. As $\nu_i : X_i \to X$ resolves the nodes $W_i \subset X$, the divisor $P_i = \nu_i^*(W_i)$ denotes preimages of these resolved nodes in X_i. Let $N_i = \nu_i^*(N)(-P_i)$. Jarvis shows in §4.1.1 [Jar98] that for each $\overline{\mathcal{R}}_i$ there exists a line bundle L_i on X_i, and a squaring map $\beta_i : L_i^{\otimes 2} \xrightarrow{\sim} N_i$ such that $\overline{\mathcal{E}}_i \simeq \nu_{i,*}L_i$ and α_i is obtained from β_i. Furthermore, he shows that $\mathrm{Aut}(\mathcal{E}_i, \alpha_i) \simeq \mathrm{Aut}(L_i, \beta_i)$.

The latter of these groups is naturally isomorphic to $\mathrm{H}^0(X_i, \mu_2)$ where $\mu_2 \subset \mathcal{O}_{X_i}^*$ is the kernel of the squaring map. Clearly $\mathrm{H}^0(X_i, \mu_2) \simeq C^0(\Gamma_i)$ in a natural way. This proves $\mathrm{Aut}_0(\xi_i) \simeq C^0(\Gamma_i)$.

Since Γ_i is connected, $\mathrm{H}^0(\Gamma_i) \simeq \{\pm 1\}$. Therefore,

$$\underline{\mathrm{Aut}}_0(\xi_i) = \mathrm{Aut}_0(\xi_i)/\{\pm 1\} \simeq C^0(\Gamma_i)/\mathrm{H}^0(\Gamma_i) \simeq \operatorname{im} \partial.$$

□

Let $\psi_i : \underline{\mathrm{Aut}}_0(\xi_i) \xrightarrow{\sim} \operatorname{im} \delta \subset C^0(\Gamma_i)$ denote the isomorphism constructed in the proof above. Let $x \in W_i$ and $a \in \mathrm{Aut}_0(\xi_i)$. In a formal neighbourhood of x we can describe the root $\overline{\mathcal{R}}_i$ as $(\overline{E}(0,0), s)$. As we showed in Remark 2.24, an automorphism of this root, in terms of standard generators, must be of the form $\left(\begin{smallmatrix} \varepsilon_1 & 0 \\ 0 & \varepsilon_2 \end{smallmatrix}\right)$ where $\varepsilon_1, \varepsilon_2 \in \{\pm 1\}$.

Lemma 4.27. *With $i > 0$ and notation above, we have $\psi_i(a)(x) = \varepsilon_1 \varepsilon_2$.*

Proof. The automorphism a can be viewed as scaling the line bundle L_i on X_i by ± 1 on each connected component of X_i. If $x_1, x_2 \in X_i$ are the two preimages of x then ε_i must be the this scaling factor on the component containing x_i. Now applying the map $\partial : C^0(\Gamma_i) \to C^1(\Gamma_i)$ finishes the proof. □

Remark 4.28. In other words, $\psi_i(a)(x) = 1$ for $x \in W_i$ iff the two generators of $\mathcal{E}_i$ at x are scaled by a with the same parity and $\psi_i(a)(x) = -1$ otherwise.

Remark 4.29. Cornalba defines inessential automorphisms to be the action of ± 1 on exceptional components of the quasi-stable models of roots (see Chapter III.2). If $\psi_i(a)(x) = -1$ then the inessential automorphism induced on the quasi-stable model is scaling by -1. This is immediate from the $\underline{\mathrm{Proj}}$ construction we used to pass from stable to quasi-stable models. In other words, the equivalence between the stable and quasi-stable models of roots identifies the two definitions of inessential automorphisms.

Now we wish to relate $\underline{\mathrm{Aut}}_0(\xi_0)$ to $C^1(\Gamma_0)$ in the same way that we related $\underline{\mathrm{Aut}}_0(\xi_i)$ to $C^1(\Gamma_i)$ for $i > 0$ in Lemma 4.26. First of all, observe that $\mathrm{Aut}_0(\xi_0)$ consists of sequences of automorphisms of $\mathcal{R}_i$ which are compatible with the synchronization. Therefore there is a natural inclusion $\mathrm{Aut}_0(\xi_0) \hookrightarrow \prod_{i=1}^m \mathrm{Aut}_0(\xi_i)$.

In addition, each of the inclusions $W_i \hookrightarrow W_0$ induces a canonical surjection $C^1(\Gamma_0) \twoheadrightarrow C^1(\Gamma_i)$. Putting all these together gives an injective map $C^1(\Gamma_0) \hookrightarrow \prod_{i=1}^m C^1(\Gamma_i)$ since each edge of Γ_0 appears in some Γ_i.

Finally, we define a map $\psi_0 : \mathrm{Aut}_0(\xi_0) \to C^1(\Gamma_0)$ as follows. Given $a \in \mathrm{Aut}_0(\xi_0)$ corresponding to $(a_i)_{i=1}^m \in \prod_{i=1}^m \mathrm{Aut}_0(\xi_i)$ and $x \in W_0 = \cup_{i=1}^m W_i$ we define $\psi_0(a)(x) = \psi_i(a_i)(x)$ for any $i > 0$ such that $x \in W_i$. Since a must respect the synchronization, this value $\psi_0(a)(x)$ is independent of the choice of i.

Lemma 4.30. *With the maps described above, the following diagram is Cartesian:*

$$\begin{tikzcd}
\mathrm{Aut}_0(\xi_0) \arrow[r, hook] \arrow[d, "\psi_0"'] \arrow[dr, phantom, "\ulcorner"] & \prod_{i=1}^m \mathrm{Aut}_0(\xi_i) \arrow[d, "\prod \psi_i"] \\
C^1(\Gamma_0) \arrow[r, hook] & \prod_{i=1}^m C^1(\Gamma_i)
\end{tikzcd}$$

Proof. The commutativity of the diagram is immediate from the definition of ψ_0. The diagram is Cartesian because the two horizontal arrows are injective. $\square$

Corollary 4.31. $\underline{\mathrm{Aut}}_0(\xi_0) \simeq \ker(C^1(\Gamma_0) \to \prod_{i=1}^m \mathrm{H}^1(\Gamma_i))$.

Proof. As in Lemma 4.26 we may identify $\underline{\mathrm{Aut}}_0(\xi_0)$ with the image of ψ_0 in $C^1(\Gamma_0)$. By Lemma 4.30 we conclude that $\underline{\mathrm{Aut}}_0(\xi_0)$ is the intersection of $C^1(\Gamma_0)$ with $\prod_{i=1}^m \underline{\mathrm{Aut}}_0(\xi_i)$ in $\prod_{i=1}^m C^1(\Gamma_i)$. Now apply Lemma 4.26. $\square$

Example 4.32. Let $X = C_1 \cup_p C_2$ where C_i are smooth irreducible curves glued together at a single point. Assume $\deg N|_{C_i}$ is odd for $i = 1, 2$ (e.g. $N = \omega_X$). Then any root of N has to be singular at the origin, by degree reasons. Pick any m-tuple of roots with a synchronization. Now $\Gamma_0 = \Gamma_1 = \cdots = \Gamma_m$ is the complete graph on two nodes and thus the first Betti number is zero. In particular, $C^1(\Gamma_i) = \ker(C^1(\Gamma_i) \to \mathrm{H}^1(\Gamma_i))$ so that we may conclude $\underline{\mathrm{Aut}}_0(\xi_0) = \{\pm 1\}$.

Example 4.33. Let $X = C/(p \sim q)$ where $(C, p, q) \in \mathcal{M}_{g,2}$. Assume once again that $N|_C$ has even degree, e.g., $N = \omega_C$. Then each root may either be singular or non-singular at the node. Pick an m-tuple of roots and define ξ_0 as usual. If all m of the roots are non-singular then $C^1(\Gamma_i) = 1$ so $\underline{\mathrm{Aut}}_0(\xi_0) = 1$. If any one of the roots is singular, say the first one, then $\Gamma_0 = \Gamma_1$ is the graph with a single vertex and a single loop, thus $\mathrm{H}^1(\Gamma_1) = \{\pm 1\}$. The map $C^1(\Gamma_0) \to C^1(\Gamma_1)$ is an isomorphism and $\ker(C^1(\Gamma_1) \to \mathrm{H}^1(\Gamma_1)) = 1$. It follows that $\underline{\mathrm{Aut}}_0(\xi_0) = 1$.

Example 4.34. Let C_1 and C_2 be smooth irreducible curves and let $p_i, q_i, r_i, s_i \in C_i$ be 4 distinct points. Let X be obtained by gluing C_1 to C_2 by the identifications $p_1 \sim p_2, \ldots, s_1 \sim s_2$. Let $p, q, r, s \in X$ stand for the corresponding nodes and suppose $N|_{C_i}$ has even degree, for instance $N = \omega_X$. This time we let $m = 2$ and we pick ξ_0 such that $W_1 = \{p, q\}$ and $W_2 = \{q, r\}$ so that $W_0 = \{p, q, r\}$. Then $\Gamma_1 = \{\{C_1, C_2\}, \{p, q\}\}$ is the graph on two vertices with two edges between them. Similarly for Γ_2. But Γ_0 is the graph on two vertices with three edges between them. For $i = 1, 2$ the maps $C^1(\Gamma_i) \to \mathrm{H}^1(\Gamma_i)$ are given by $(\varepsilon_1, \varepsilon_2) \mapsto \varepsilon_1\varepsilon_2$. Therefore $\underline{\mathrm{Aut}}_0(\xi) = \ker(\{\pm 1\}^3 \to \{\pm 1\}^2 : (\varepsilon_1, \varepsilon_2, \varepsilon_3) \mapsto (\varepsilon_1\varepsilon_2, \varepsilon_2\varepsilon_3))$. In conclusion, $\underline{\mathrm{Aut}}_0(\xi) = \{\pm(1, 1, 1)\}$.

Remark 4.35. Note that in the last example, all four possibilities are realized: The first root is singular at p but not the second root. The second root is singular at r but not the first. Both roots are singular at q and both are non-singular at s. This example is one of the reasons why the definition of a synchronization had to be so complicated.

4.5 Automorphisms of the deformation functor

For convenience, we will make the following identifications $\xi := \xi_0$, $\Gamma := \Gamma_0$, $W := W_0$ and $\psi := \psi_0$. Let $x_1, \ldots, x_n \in X$ be the nodes of X indexed so that there is an integer $n' \leq n$ such that at least one root $\mathcal{R}_i \in \mathcal{R}$ is singular at x_j iff $j \leq n'$.

Let D_ξ be the functor of infinitesimal deformations of ξ. Recall that we are given a moduli space $\mathcal{M}$ in which (X, N) deforms. Let us denote by $D_{(X,N)}$ the infinitesimal deformations of (X, N) over $\mathcal{M}$.

Notation 4.36. Denote by $\mathrm{Aut}(D_\xi)$ the automorphisms of the functor D_ξ and by $\mathrm{Aut}_0(D_\xi) \subset \mathrm{Aut}(D_\xi)$ the automorphisms inducing the identity on $D_{(X,N)}$.

We described in the previous section that $D_{(X,N)}$ is pro-represented by a ring T which contains elements $\tilde{t}_i$ for $i = 1, \ldots, n$ such that $\tilde{t}_i = 0$ corresponds to the locus where the node x_i remains singular. Then D_ξ is pro-represented by a ring T' such that the forgetful map $D_\xi \to D_{(X,N)}$ identifies T' with $T[\tau_1, \ldots, \tau_{n'}]/(\tau_1^2 - \tilde{t}_1, \ldots, \tau_{n'}^2 - \tilde{t}_{n'})$.

Fix a universal family over $\mathrm{Spec}(T')$ and identify the functors D_ξ and $\hom(\_, T')$. Similarly identify $\mathrm{Spec}(T)$ with $D_{(X,N)}$. These identifications give an isomorphism $\mathrm{Aut}_0(D_\xi) \simeq \mathrm{Aut}_T(T')$. But any automorphism of T' over T must be of the form $(\tau_1, \ldots, \tau_{n'}) \mapsto (\varepsilon_1\tau_1, \ldots, \varepsilon_{n'}\tau_{n''})$ where $\varepsilon_i \in \{\pm 1\}$ for each i. Conversely, any map of this form will belong to $\mathrm{Aut}_T(T')$. The indices of τ correspond to the nodes of X, and hence to the edges of the graph Γ. It is clear that the choice of a different universal family of $\mathrm{Spec}(T)$ or $\mathrm{Spec}(T')$ will not change this identification.

Notation 4.37. Let $\sigma : \mathrm{Aut}_0(D_\xi) \xrightarrow{\sim} C^1(\Gamma)$ denote this natural isomorphism.

There is a natural action of $\mathrm{Aut}(\xi)$ on D_ξ inducing a group homomorphism $\mathrm{Aut}(\xi) \to \mathrm{Aut}(D_\xi)$. As this is standard, we will briefly sketch how this works. Let $(f, \varphi) \in \mathrm{Aut}(\xi)$ and $\mathbb{X} = (\iota : X \hookrightarrow \mathcal{X}, \mathfrak{j} : \mathcal{R} \twoheadrightarrow \overline{\mathcal{R}}, \Phi) \in D_\xi$ be an infinitesimal deformation of ξ over some base. Then we can define the following deformation $\mathbb{X}' = (\iota \circ f^{-1}, \varphi \circ \mathfrak{j}, \Phi) \in D_\xi$. This defines an automorphism $D_\xi \to D_\xi : \mathbb{X} \mapsto \mathbb{X}'$. Observe that if $f = \mathrm{id}_X$ then the resulting action on $D_{(X,N)}$ is trivial.

Notation 4.38. Denote by $\rho : \mathrm{Aut}_0(\xi) \to \mathrm{Aut}_0(D_\xi)$ the restriction of the morphism $\mathrm{Aut}(\xi) \to \mathrm{Aut}(D_\xi)$ defined above to $\mathrm{Aut}_0(\xi)$ with appropriately modified codomain.

To compute the singularities of the coarse moduli space of roots at the isomorphism class of ξ the first step is to compute the action of $\mathrm{Aut}_0(\xi)$ on T'. In other words, we wish to describe the image of the map $\sigma \circ \rho$. The following theorem does precisely that.

Theorem 4.39. *The following diagram commutes:*

$$\begin{tikzcd} \mathrm{Aut}_0(\xi) \arrow[rr, "\rho"] \arrow[dr, "\psi"'] & & \mathrm{Aut}_0(D_\xi) \arrow[dl, "\simeq"', "\sigma"] \\ & C^1(\Gamma) & \end{tikzcd}$$

Proof. Pick $x_j \in W$ and $a = (a_i)_{i=1}^m \in \mathrm{Aut}_0(\xi)$. We wish to show $\psi(a)(x) = \sigma \circ \rho(a)$. Pick $i \in \{1, \ldots, m\}$ such that $x_j \in W_i \subset W$, i.e., the i-th root is singular at x_j. Lemma 4.27 implies that $\psi(a)(x_j) = \psi_i(a_i)(x_j) = \varepsilon_1\varepsilon_2$ where a_i is formal locally of the form $\left(\begin{smallmatrix} \varepsilon_1 & 0 \\ 0 & \varepsilon_2 \end{smallmatrix}\right)$ around x_j. On the other hand, $\sigma \circ \rho(a)(\tau_j) = c_j\tau_j$ for some $c_j \in \{\pm 1\}$. Therefore, we aim to prove $c_j = \varepsilon_1\varepsilon_2$.

Let $(X \overset{\iota}{\hookrightarrow} \mathcal{X}, \mathfrak{j} : \mathcal{R} \to \overline{\mathcal{R}}, \Phi)$ be the *universal* deformation of ξ over T'. Identify x_j with $\iota(x_j)$ and let $U_j = \mathrm{Spec}(\hat{\mathcal{O}}_{\mathcal{X},x_j}) \to \mathcal{X}$ be the formal neighbourhood of x_j in $\mathcal{X}$. Using Theorem 4.21, Lemma 4.20 and Theorem 2.30 we conclude $\mathcal{R}_i|_{U_j} \simeq (E(\tau_j, \tau_j), s)$.

The automorphism $\rho(a)$ maps the universal deformation to $(\iota, a \circ \mathfrak{j}, \Phi)$. In other words, there exists an isomorphism $l : \mathcal{R} \xrightarrow{\sim} \rho(a)^*\mathcal{R}$, necessarily unique, such that the composition below coincides with a:

$$\iota^*\mathcal{R} \xrightarrow{\iota^* l} \iota^*\rho(a)^*\mathcal{R} \xrightarrow{\sim} \iota^*\mathcal{R},$$

where the last isomorphism is the natural one coming from the identity $\iota = \rho(a) \circ \iota$. Let $l = (l_i)_{i=1}^m$ with $l_i : \mathcal{R}_i \xrightarrow{\sim} \mathcal{R}_i$.

Restrict l_i to U_j and notice that $\rho(a)^*(E(\tau_j, \tau_j), s) \simeq (E(c_j\tau_j, c_j\tau_j), s)$. We emphasize that since $\tau_j \neq 0$ this isomorphism is unique up to sign as a result of Lemma 2.22. Since l and a must agree in the sense above, $l_i|_{U_j} : (E(\tau_j, \tau_j), s) \xrightarrow{\sim} (E(c_j\tau_j, c_j\tau_j), s)$ must restrict to a_i near x_j. We apply Remark 2.24 to conclude that the matrix form of $l_i|_{U_j}$ is completely determined on the fiber over x_j. Since l_i and a_i agree there, we have $l_i|_{U_j} = \left(\begin{smallmatrix} \varepsilon_1 & 0 \\ 0 & \varepsilon_2 \end{smallmatrix}\right)$. Applying Lemma 2.22 to $l_i|_{U_j}$ gives $c_j = \varepsilon_1\varepsilon_2$. $\square$

Chapter 5

The moduli space of roots

5.1 Single roots

Recall the definition of $\mathcal{S}(\mathcal{N})$ and $\overline{\mathcal{S}}(\mathcal{N})$ from Definition 1.11 together with our conventions on $\mathcal{M} \to S$ and $\mathcal{C} \to \mathcal{M}$ from Section 1.2.

5.1.1 Fundamental properties of the moduli space of roots

Here we list the basic results that can be obtained from Jarvis' work on spin curves [Jar98]. We simply point to the relevant results and make a few remarks that may be particular to our situation.

Proposition 5.1. $\overline{\mathcal{S}}(\mathcal{N})$ *is an algebraic stack.*

Proof. We used the hypothesis that $\mathcal{N}$ has absolutely bounded degree to guarantee the result of Sublemma 4.1.10 [Jar98]. The rest of the proof of Proposition 4.1.7 [Jar98] works without requiring that $\mathcal{N}$ be the dualizing sheaf and providing us with a smooth cover.

The proof of Proposition 4.1.14 makes no use of the fact that $\mathcal{N}$ is the dualizing sheaf and applies equally well to our case, proving that the diagonal $\overline{\mathcal{S}}(\mathcal{N}) \to \mathcal{M}$ is representable. Since $\mathcal{M} \to S$ has representable diagonal, we are done. $\square$

Proposition 5.2. *The diagonal of* $\overline{\mathcal{S}}(\mathcal{N}) \to \mathcal{M}$ *is finite.*

Proof. The diagonal is representable and hence of finite type. The diagonal is also unramified by Proposition 4.1.15 [Jar98], whose arguments apply without change to our situation.

To finish the proof we need to show that the diagonal is proper. This is done using the valuative criterion and working with a complete DVR as the base.

In this case the automorphisms come in two flavors as described in Lemma 2.22. Either the automorphism is multiplication by ± 1 or for each node that persists over the entire base we have an additional automorphism of order 2.

There is no obstacle to extending these additional automorphisms as they are also defined by ± 1. Which means that any automorphism on the generic fiber extends to the special fiber. $\square$

Proposition 5.3. *$\overline{\mathcal{S}}(\mathcal{N}) \to \mathcal{M}$ is proper and of finite type.*

Proof. The proof of Lemma 4.1.8 [Jar98] constructs a smooth atlas of $\overline{\mathcal{S}}(\mathcal{N})$ and it is easy to see that this atlas is of finite type over the curve $\mathcal{C} \to \mathcal{M}$.

The properness of this map follows from §4.2.2 *loc.cit.* when the fibers of $\mathcal{C} \to \mathcal{M}$ are generically smooth. When $\mathcal{C} \to \mathcal{M}$ consists entirely of singular curves we can do the following.

Let R be a DVR with quotient field K. Suppose we have a commutative diagram

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & \overline{\mathcal{S}}(\mathcal{N}) \arrow[d] \\ \operatorname{Spec} R \arrow[r] & \mathcal{M} \end{tikzcd}.$$

Let $(C_R \to \operatorname{Spec} R, \mathcal{N}_R)$ be obtained from the bottom horizontal map and $(\mathcal{E}_K, b_K : \mathcal{E}_K^2 \to \mathcal{N}_K)$ be the root obtained from the top horizontal map. Normalize the curve $C_R \to \operatorname{Spec} R$. For notational convenience we will assume there are only two components in this normalization, say Y_1, Y_2, with $Z_i \subset Y_i$ glued together to form C_R. Let $z_i = Y_i|_K \cap Z_i$.

Pulling back $(\mathcal{E}_K, b_K)$ to $Y_i|_K$ gives a root of $\mathcal{N}|_{Y_i|_K}(-z_i)$, see §4.1.4.1 [Jar98] for more about roots on curves over a field. Then, applying §4.2.2 *loc.cit.* to each of these components but using the twisted line bundles $\mathcal{N}|_{Y_i}(-Z_i)$, we can extend our root to these components (possibly after a finite base change). Note that this extension is unique.

Let $(\mathcal{E}_i, b_i)$ be the root of $\mathcal{N}|_{Y_i}(Z_i)$ on Y_i obtained by extending $(\mathcal{E}_K, b_K)|_{Y_i|_K}$. Let $\pi_i : Y_i \to C_R$ be the inclusion maps. The torsion-free sheaf underlying our root will be $\mathcal{E} := \pi_{1,*}\mathcal{E}_1 \oplus \pi_{2,*}\mathcal{E}_2$.

Each b_i induces a map $b_i' : \pi_{i,*}b_i : \pi_{i,*}\mathcal{E}_i^2 \to \pi_{i,*}\mathcal{N}|_{Y_i}(-Z_i) \hookrightarrow \mathcal{N}$ so that we may let $b : \mathcal{E}^2 \to \mathcal{N}$ be the map which agrees with b_i' for $i = 1, 2$. Note that b is the zero map along Z.

We need only show that this root is unique to finish the proof. This follows from the classification of roots near singularities. Since R is integral, the singularity is of the form $R[[x, y]]/(xy)$. Then the only roots around this singularity are of the form $(E(0, 0), s)$. This agrees with our construction. □

Remark 5.4. Jarvis claims that the moduli space of r-th roots is *finite* over the moduli space of curves. This is not true since each root has a non-trivial automorphism over the curve (namely multiplication by constants whose r-th power is 1). Therefore this morphism is not representable, hence not finite.

Definition 5.5. If a morphism $f : \mathcal{X} \to \mathcal{Y}$ of algebraic stacks has unramified diagonal $\Delta_f : \mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ then let us call $\mathcal{X}$ a *relatively DM stack over* $\mathcal{Y}$. The morphism f is then called a DM stack.

Remark 5.6. In the footnote 1 of [Stacks, Tag 04YV] it is shown that the fibers over schemes of a relatively DM stack are DM stacks.

Corollary 5.7. *$\overline{\mathcal{S}}(\mathcal{N}) \to S$ is a Deligne–Mumford stack if $\mathcal{M}$ is a Deligne–Mumford stack.*

Proof. Since $\overline{\mathcal{S}}(\mathcal{N}) \to \mathcal{M}$ has unramified diagonal, $\overline{\mathcal{S}}(\mathcal{N}) \to \mathcal{M}$ is relatively DM by definition. Being relatively DM is stable under composition and being relatively DM over a scheme is equivalent to being DM. See [Stacks, Tag 04YV] for proofs of these statements. ☐

Proposition 5.8. *If the natural map $\mathcal{M} \to \overline{\mathcal{M}}_g$ is smooth then $\overline{\mathcal{S}}(\mathcal{N}) \to S$ is smooth.*

Proof. We have not defined multiple roots for families but when $m = 1$ we can still use Corollary 4.23 from which the result is immediate. ☐

Proposition 5.9. $\overline{\mathcal{S}}(\mathcal{N}) \to \mathcal{M}$ *is quasi-finite.*

Proof. See §4.1.4.1 of [Jar98] where the isomorphism classes of roots on a curve over a field are identified with isomorphism classes of line bundles squaring to a fixed line bundle on partial normalizations. The latter set is finite hence we are done. ☐

5.1.2 Basic properties regarding families of roots

Given a family of curves $C \to T$, the individual singularities of each fiber are referred to as the nodes of a fiber. But the locus of these nodes are a well defined closed subscheme of C and we use the following terminology to distinguish this locus from individual nodes.

Definition 5.10. For a family of nodal curves $C \to T$, the closed subscheme $Z \subset C$ cut out by the first fitting ideal of the relative dualizing sheaf $\Omega_{C/T}$ is called the *discriminant locus of the curve C/T.*

Remark 5.11. If $T' \to T$ and $C' = C \times_T T' \to T'$ is the pullback curve then the discriminant locus Z' of the curve C'/T' coincides with the pullback $Z \times_C C'$ of the discriminant locus Z of C/T. For this reason we say the discriminant locus behaves well with respect to base change.

Recall that the torsion-free module underlying a root is locally self-dual. The following observation is useful in visualizing where the difficulties lie in trying to synchronize a pair of roots: it is never an arbitrary part of the discriminant locus but a union of connected components of it.

Lemma 5.12. *Let $C \to T$ be a stable curve and $\mathcal{E}$ a locally self-dual rank-1 torsion-free module on the curve. Then the rank of $\mathcal{E}$ is constant on each component of the discriminant locus.*

Proof. Pick any point $\mathfrak{p} \in Z$. The module $\mathcal{E}$ has either rank 1 or rank 2 at $\mathfrak{p}$. We will show both the properties of being rank 1 and rank 2 are open in Z around $\mathfrak{p}$. By semi-continuity, being of rank 1 is an open condition so it remains to show the latter.

First perform an étale base change if necessary, and then pass to an étale neighbourhood $U \to C$ of $\mathfrak{p}$ so that may assume $B = \operatorname{Spec} R$ and $U = \operatorname{Spec} A$ where $\exists x, y \in A$, $\pi \in R$ such that Z_U is the vanishing set of (x, y) and $\pi = xy$. See Remark 5.39.

By choosing U appropriately, we may apply Faltings' classification [Fal96] and conclude that $E := \mathcal{E}_U$ is either free or of the form $E(p, p)$ where $p^2 = \pi$. Note that being free is an open condition, so that we need only consider the latter condition.

Note Z_U is isomorphic to $\operatorname{Spec} R/(\pi)$ and $p^2 = \pi$. Now $E(p,p)$ is free at a point of Z_U iff p is invertible at that point. However, this is impossible in $R/(\pi)$ since $p^2 = 0 \mod (\pi)$. This proves that being of rank 2 (along the node) is open around $\mathfrak{p}$. □

5.2 Multiple roots

Let T be a scheme, $C \to T$ a stable curve and $\mathcal{N}$ a line bundle on C. Let $\mathcal{R} = (\mathcal{E}_i, b_i)_{i=1}^m$ a sequence of roots of $\mathcal{N}$.

Definition 5.13. For each $T' \to T$, and for each $i = 1, \dots, m$, define $V_i(T') \subset C' := C \times_T T'$ to be the locus of points x where the rank of $\mathcal{E}_i$ is maximal among all $\mathcal{E}_j$. In symbols, $x \in V_i(T')$ iff

$$\dim_{\kappa(x)} \mathcal{E}_i|_x = \max\{\dim_{\kappa(x)} \mathcal{E}_j|_x \mid j = 1, \dots, m\}.$$

Remark 5.14. In light of Lemma 5.12 the locus $V_i(T')$ is open: it is the complement of a finite number of components of the discriminant locus, where $\mathcal{E}_i$ is free but some $\mathcal{E}_j$ are not.

We will first state the definition of a multiple root which has the advantage of capturing the geometric interpretation most readily. See Remark 5.16 for a summary of this interpretation. In Section 5.2.1 we will state and prove alternative definitions relating it to our previous work.

Definition 5.15 (Multiple Root)**.** Let D be a sheaf of graded algebras on C and let $\Psi = (\psi_i : D \to \operatorname{Sym}^{2*} \mathcal{E}_i)_{i=1}^m$ be a sequence of graded morphisms such that

- for all i, j we have $\operatorname{Sym} b_i \circ \psi_i = \operatorname{Sym} b_j \circ \psi_j$,
- for all i the map $\psi_i|_{V_i}$ is an isomorphism (see Definition 5.13).

Then the tuple $(\mathcal{R}, \Psi)$ will be called a *multiple root*.

Remark 5.16. The motivation for this definition can be summarized as follows. Each root $\mathcal{E}_i$ gives rise to a space $\mathbb{P}(\mathcal{E}_i) \to C$ together with a line bundle $\mathcal{L}_i$ on $\mathbb{P}(\mathcal{E}_i)$ which pushes forward to $\mathcal{E}_i$ on C. The definition above identifies these $\mathbb{P}(\mathcal{E}_i)$ wherever possible as well as the line bundles $\mathcal{L}_i^{\otimes 2}$, when this makes sense. For a more comprehensive treatment, see Chapter III.2.

Remark 5.17. Note that D is locally generated in degree 1, because $\operatorname{Sym}^{2*} \mathcal{E}_i$ are locally generated in degree 1.

Let $\mathcal{M} \to S$ be a DM stack, locally of finite type over an excellent scheme $S \to \operatorname{Spec} \mathbb{Z}[1/2]$. Let $\mathcal{C} \to \mathcal{M}$ be a stable curve over $\mathcal{M}$ and $\mathcal{N}$ a line bundle on $\mathcal{C}$.

Definition 5.18. Let $\overline{\mathcal{S}}^m(\mathcal{N})$ be the category fibered in groupoids defined over $\mathcal{M}$ whose fiber over $T \to \mathcal{M}$ is the groupoid of multiple roots of $\mathcal{N}_B$ on $\mathcal{C}_B := \mathcal{C} \times_{\mathcal{M}} B$.

Now we give a definition that is more convenient to study general properties of $\overline{\mathcal{S}}^m(\mathcal{N})$.

5.2.1 Working definition for multiple roots

The symmetric algebra generated by the roots is not a finitely presented module, which makes it inconvenient to work with. We will now rectify this problem.

Definition 5.19. Let $\mathcal{F}$ be a sheaf of modules on C and let $\Phi := (\varphi_i : \mathcal{F} \to \mathcal{E}_i^2)_{i=1}^m$ be a sequence of maps such that

- for all i, j we have $b_i \circ \varphi_i = b_j \circ \varphi_j$,
- for all i the map $\varphi_i|_{V_i}$ is an isomorphism.

Then we will call Φ a *pre-sync data* and the pair $(\mathcal{R}, \Phi)$ will be called a *pre-synced tuple of roots.* We will refer to these two conditions as *pre-sync conditions.*

Remark 5.20. The maps $b_i : \mathcal{E}_i^2 \to \mathcal{N}$ glue to a map $b_\Phi : \mathcal{F} \to \mathcal{N}$.

On $V_{ij} := V_i \cap V_j$ we can define $\psi_{ij} = \varphi_j|_{V_{ij}} \circ \varphi_i|_{V_{ij}}^{-1} : \mathcal{E}_i^2 \xrightarrow{\sim} \mathcal{E}_j^2$. Using ψ_{ij}, we get a surjective map

$$\operatorname{Sym}^* \operatorname{Sym}^2 \mathcal{E}_i|_{V_{ij}} \to \operatorname{Sym}^{2*} \mathcal{E}_j|_{V_{ij}}.$$

We would like this to factor through an isomorphism $\operatorname{Sym}^{2*} \mathcal{E}_i|_{V_{ij}} \xrightarrow{\sim} \operatorname{Sym}^{2*} \mathcal{E}_j|_{V_{ij}}$, we give it a name when this happens.

Definition 5.21. If for each tuple i, j the isomorphism ψ_{ij} induces an isomorphism $\operatorname{Sym}^{2*} \mathcal{E}_i|_{V_{ij}} \xrightarrow{\sim} \operatorname{Sym}^{2*} \mathcal{E}_j|_{V_{ij}}$ then we say Φ is a *sync data.* The tuple $(\mathcal{R}, \Phi)$ is then called a *synced tuple of roots.* This condition will be called the *sync condition.*

Lemma 5.22. *The sync condition holds whenever the morphism*

$$\operatorname{Sym}^2 \operatorname{Sym}^2 \mathcal{E}_i|_{V_{ij}} \to \operatorname{Sym}^4 \mathcal{E}_j|_{V_{ij}}$$

factors through $\operatorname{Sym}^4 \mathcal{E}_i|_{V_{ij}} \to \operatorname{Sym}^4 \mathcal{E}_j|_{V_{ij}}$.

Proof. This follows because the kernel of $\operatorname{Sym}^* \operatorname{Sym}^2 \mathcal{E}_i \to \operatorname{Sym}^{2*} \mathcal{E}_i$ is generated by the kernel of $\operatorname{Sym}^2 \operatorname{Sym}^2 \mathcal{E}_i \to \operatorname{Sym}^4 \mathcal{E}_i$. □

For a pair of pre-synced or synced roots the definition of an isomorphism is the natural one: a sequence of isomorphisms between the roots which lift to D's or $\mathcal{F}$'s respectively. We define only one here.

Definition 5.23. An isomorphism between a pair of synced roots $\mathfrak{R} = (\mathcal{R}, \Phi)$ and $\mathfrak{R}' = (\mathcal{R}, \Phi')$ is a sequence of isomorphisms $\mu = (\mu_i)_{i=1}^m : \mathcal{R} \to \mathcal{R}'$ of the underlying roots, admitting an isomorphism $\mu_0 : \mathcal{F} \xrightarrow{\sim} \mathcal{F}'$ commuting with the synchronizations.

Remark 5.24. Notice that, whenever the morphism μ_0 exists, it is uniquely defined by $\mu = (\mu_i)_{i=1}^m$.

Lemma 5.25. *The category of synced tuples of roots on C and the category of multiple roots are equivalent.*

Proof. Starting from $(\psi_i : D \to \operatorname{Sym}^{2*} \mathcal{E}_i)$ we define $\mathcal{F} := D_1$, $\varphi_i := \psi_{i,1} : \mathcal{F} \to \mathcal{E}_i^2$. This is clearly a pre-sync data and to see that this is a sync data we need only recall that D is locally generated in degree 1 and that $D_2|_{V_i} \xrightarrow{\sim} \operatorname{Sym}^4 \mathcal{E}_i|_{V_i}$.

For the converse, we first need to define D. This is done by gluing together $\operatorname{Sym}^{2*} \mathcal{E}_i|_{V_i}$ on each V_i. To perform this gluing we note that ψ_{ij} induces an isomorphism between the symmetric algebras by the definition of sync data. The cocyle condition is satisfied because ψ_{ij} is locally of the form $\varphi_j \circ \varphi_i^{-1}$.

To define the maps $\psi_i : D \to \operatorname{Sym}^{2*} \mathcal{E}_i$, we begin with the isomorphisms $D|_{V_i} \xrightarrow{\sim} \operatorname{Sym}^{2*} \mathcal{E}_i|_{V_i}$. For any p where $\mathcal{E}_i$ is free, e.g. for any $p \notin V_i$, find j such that $p \in V_j$ and choose an open neighbourhood U of p such that $U \subset V_j$ and $\mathcal{E}_i$ is free on U. Then we observe that b_i, and $\operatorname{Sym} b_i$, are isomorphisms on U. So we may now define

$$D|_U \to \operatorname{Sym}^{2*} \mathcal{E}_j|_U \xrightarrow{b_j} \operatorname{Sym}^* \mathcal{N} \overset{b_i^{-1}}{\xrightarrow{\sim}} \operatorname{Sym}^{2*} \mathcal{E}_i|_U.$$

Because $b_j \circ \varphi_j = b_k \circ \varphi_k$, the map $D|_U \to \operatorname{Sym}^{2*} \mathcal{E}_i|_U$ is independent of our choice of j. In particular, these maps glue together to give $\psi_i : D \to \operatorname{Sym}^{2*} \mathcal{E}_i$. □

5.3 Algebraicity

Our goal in this section is to show that $\overline{\mathcal{S}}^m(\mathcal{N})$ is an algebraic stack and establish its basic properties. We know that $\overline{\mathcal{S}}(\mathcal{N})$ is an algebraic stack (see Proposition 5.1) and we will prove that $\overline{\mathcal{S}}^m(\mathcal{N})$ is an algebraic stack by induction on $m \geq 1$. We now prepare to make this induction step possible.

Lemma 5.26. *There is a canonical forgetful functor $\overline{\mathcal{S}}^m(\mathcal{N}) \to \overline{\mathcal{S}}^{m-1}(\mathcal{N})$ forgetting the m-th root and adjusting the sync data appropriately.*

Proof. Let $C \to T$ be a stable curve, with $\mathcal{R} = (\mathcal{E}_i, b_i)_{i=1}^m$ and $\Phi = (\varphi_i : \mathcal{F} \to \mathcal{E}_i^2)_{i=1}^m$ giving a synchronized tuple of roots. We want to define a sync data $\Phi' = (\varphi_i' : \mathcal{F}' \to \mathcal{E}_i^2)_1^{m-1}$ for the tuple $\mathcal{R}' := (\mathcal{E}_i, b_i)_{i=1}^{m-1}$.

Recall that for each $i = 1, \ldots, m$ we defined V_i to be the locus in C for which $\mathcal{E}_i$ has maximal rank among the m roots. Similarly, for each $i = 1, \ldots, m-1$ define W_i to be the locus in C for which $\mathcal{E}_i$ has maximal rank among the first $m-1$ roots. Clearly $V_i \subset W_i$.

We will define $\mathcal{F}'$ by gluing $\mathcal{E}_i^2|_{W_i}$ together for $i = 1, \ldots, m-1$. Let $Z_{ij} \subset C$ be the closed locus in which $\mathcal{E}_i$ and $\mathcal{E}_j$ are both non-free. Clearly $Z_{ij} \subset V_i \cap V_j \subset W_i \cap W_j$. Outside of Z_{ij} the maps b_i and b_j are isomorphisms, so the isomorphism $\varphi_{ij} : \mathcal{E}_i^2|_{V_{ij}} \xrightarrow{\sim} \mathcal{E}_i^2|_{V_{ij}}$ naturally extends to all of $W_{ij} = W_i \cap W_j$ as $b_j^{-1} \circ b_i$. This gluing procedure defines $\varphi_i : \mathcal{F}' \to \mathcal{E}_i^2$ for all $i = 1, \ldots, m-1$ and it is easy to check that this is a sync data. □

Remark 5.27. For any proper subset $J \subset \{1, \ldots, m\}$ we can forget the roots indexed by J and modify the synchronizations appropriately, however we will not use this except for the simple case where $J = \{1, \ldots, m-1\}$. This defines a map $\overline{\mathcal{S}}^m(\mathcal{N}) \to \overline{\mathcal{S}}(\mathcal{N})$ forgetting all but the m-th root. Now combine the two maps, the first map forgetting the m-th root and the second map forgetting all roots except the m-th root. We obtain a map

$\overline{\mathcal{S}}^m(\mathcal{N}) \to \overline{\mathcal{S}}^{m-1}(\mathcal{N}) \times_{\mathcal{M}} \overline{\mathcal{S}}(\mathcal{N})$. In effect, this map forgets how the m-th root is synchronized with the rest of the roots.

Take $B \to \overline{\mathcal{S}}^{m-1}(\mathcal{N}) \times_{\mathcal{M}} \overline{\mathcal{S}}(\mathcal{N})$ and denote the corresponding data by $(C \to T, \mathcal{N}, \mathfrak{R}', \mathcal{R}_m)$ where $\mathfrak{R}' = ((\mathcal{R}_i)_{i=1}^{m-1}, \Phi' = (\varphi_i' : \mathcal{F} \to \mathcal{E}_i))$ is an $m-1$ tuple of synced roots of $\mathcal{N}$, and $\mathcal{R}_m$ a root of $\mathcal{N}$. As usual $\mathcal{R}_i = (\mathcal{E}_i, b_i)$.

Let $\mathcal{F}$ be a coherent sheaf of modules on C and let $\tau_1 : \mathcal{F} \to \mathcal{F}'$, $\tau_2 : \mathcal{F} \to \mathcal{E}_m^2$ be morphisms. Define $\varphi_i : \mathcal{F} \to \mathcal{E}_i^2$ by $\varphi_i = \varphi_i' \circ \tau_1$ if $i < m$ and by $\varphi_m = \tau_2$ otherwise. Denote the m-tuple of roots by $\mathcal{R} = (\mathcal{R}_i)_{i=1}^m$.

Definition 5.28. If Φ is a (pre-)sync data for $\mathcal{R}$ then we will call (τ_1, τ_2) a *(pre-)sync data for $\mathfrak{R}'$ and $\mathcal{R}_m$.*

Theorem 5.29. $\overline{\mathcal{S}}^m(\mathcal{N})$ *is an algebraic stack, locally of finite type over S.*

Proof. We will use induction on m. The base case $m = 1$ is Proposition 5.1 and Proposition 5.3 so we assume $m \geq 2$.

Let $\mathcal{Y} = \overline{\mathcal{S}}^{m-1}(\mathcal{N}) \times_{\mathcal{M}} \overline{\mathcal{S}}(\mathcal{N})$ and consider the forgetful map $\overline{\mathcal{S}}^m(\mathcal{N}) \to \mathcal{Y}$ described in Remark 5.27.

By induction hypothesis, $\mathcal{Y}$ is an algebraic stack, locally of finite type over S, and so we need to show that for any $\mathcal{Y}$-scheme B the pullback $\overline{\mathcal{S}}^m(\mathcal{N})_B \to B$ is an algebraic stack locally of finite over B. We will do this by carving out $\overline{\mathcal{S}}^m(\mathcal{N})_B$ from an ambient stack.

The ambient stack: Take $B \to \mathcal{Y}$. This defines a curve $C \to B$ and roots $(\mathcal{E}_i, b_i)_{i=1}^{m-1}$ together with a sync data $\Phi' = (\varphi_i' : \mathcal{F}' \to \mathcal{E}_i^2)_{i=1}^{m-1}$. In addition we have a root $(\mathcal{E}_m, b_m)$ on C. For any $B' \to B$ denote by $\mathcal{F}'_{B'}$ the pullback of $\mathcal{F}'$ to $C_{B'} := C \times_B B'$, similarly define $\varphi_i|_{B'}$ and $\mathcal{E}_i|_{B'}$. Denote by $b' : \mathcal{F}' \to \mathcal{N}_B$ the pullback of the map $b_{\Phi'} : \mathcal{F} \to \mathcal{N}$.

Let $\mathcal{A}' \to B$ be the category of tuples $(B' \to B, \mathcal{F})$ where $\mathcal{F}$ is a quasi-coherent sheaf on $C_{B'}$ which is B'-flat, finitely presented and has B'-proper support. In [Hal] it is shown that $\mathcal{A}'$ is an algebraic stack, locally of finite type over B.

Let $\mathcal{A} \to \mathcal{A}'$ be the category of tuples $(B' \to B, \mathcal{F}, (\tau_1 : \mathcal{F} \to \mathcal{F}'_{B'}, \tau_2 : \mathcal{F} \to \mathcal{E}_m|_{B'}))$ with $(B' \to B, \mathcal{F}) \in \mathcal{A}'$ and $b'|_{B'} \circ \tau_1 = b_m|_{B'} \circ \tau_2$. Since all the relevant modules are finitely presented, $\mathcal{A}$ is an algebraic stack and will serve as our ambient stack.

It is clear that $\overline{\mathcal{S}}^m(\mathcal{N})$ is a subcategory of $\mathcal{A}$ and we will now show that the inclusion $\overline{\mathcal{S}}^m(\mathcal{N}) \to \mathcal{A}$ is a locally closed immersion.

Let $\mathcal{A}_1 \subset \mathcal{A}$ be the subcategory of $\mathcal{A}$ for which τ_i's satisfy the pre-sync condition. Let $\mathcal{A}_2 \subset \mathcal{A}_1$ be the subcategory of $\mathcal{A}_1$ for which the τ_i's satisfy the sync condition. Note $\mathcal{A}_2 = \overline{\mathcal{S}}^m(\mathcal{N})$.

We are done once we prove that $\mathcal{A}_1 \to \mathcal{A}$ is an open immersion and $\mathcal{A}_2 \to \mathcal{A}_1$ a closed immersion.

Pre-sync condition is open: The proof that $\mathcal{A}_1 \to \mathcal{A}$ is an open immersion takes all of Section 5.3.1 culminating in Proposition 5.34.

Sync condition is closed: The proof that $\mathcal{A}_2 \to \mathcal{A}_1$ is a closed immersion takes place in Section 5.3.2, see Proposition 5.35. $\square$

5.3.1 Pre-sync condition is open

Definition 5.30. For any $T \to \mathcal{A}$ we will define $U_i(T), V_i(T) \subset C_T$ for $i = 1, 2$ as follows. Let $U_i(T) = \{x \in C_T \mid \tau_i|_x \text{ is an isomorphism}\}$. Let $V_1(T)$ be the complement of the loci where $\mathcal{F}'$ is free but $\mathcal{E}_m$ is not. Let $V_2(T)$ be the complement of the loci where $\mathcal{E}_m$ is free but $\mathcal{F}'$ is not.

Remark 5.31. Whenever we are working locally on C, we may assume $m = 2$ since $\mathcal{F}'$ is locally isomorphic to one of $\mathcal{E}_i^2$. When $m = 2$ then $\mathcal{F}' = \mathcal{E}_1^2$.

With this remark in mind we will assume that there is a pair of maps $(\varphi_i : \mathcal{F} \to \mathcal{E}_i^2)_{i=1}^2$ satisfying $b_1 \circ \varphi_1 = b_2 \circ \varphi_2$ and that our goal is to show the second pre-sync condition defines an open locus on the base T. Note that the second pre-sync condition is equivalent to having $U_i = V_i$.

Lemma 5.32. *The sets U_i and V_i are open. Furthermore, they respect base change. More precisely, for any $S \to T \to \mathcal{A}$ we have:*

- $V_i(S) = V_i(T)|_S$
- $U_i(S) = U_i(T)|_S$

Proof. The complement of V_i is the locus of points for which $\mathcal{E}_i$ is free but $\mathcal{E}_j$ is not ($j \neq i$). This locus is supported on the discriminant locus. But we showed in Lemma 5.12 that the rank of a root is constant on each connected component of the discriminant locus. Thus V_i^c is a union of components of the discriminant locus, which is closed.

Moreover, the condition of being locally free or not behaves well with respect to base change. Therefore it is clear that $V_i(S) = V_i(T)|_S$.

The fact that the U_i's respect base change and are open follows from a general fact. Let $\psi : F \to E$ be a map of finitely presented modules on C_T. Then the set where ψ is an isomorphism is the intersection $\{\ker \psi = 0\} \cap \{\operatorname{coker} \psi = 0\}$. When E is flat over T then for any $T' \to T$ we have $\{(\ker \psi)|_{T'} = 0\} \cap \{(\operatorname{coker} \psi)|_{T'} = 0\} = \{\ker(\psi|_{T'}) = 0\} \cap \{\operatorname{coker}(\psi|_{T'}) = 0\}$.

It is a standard fact that the zero set of a finitely generated module is open and respects base change. Hence we are done. □

Lemma 5.33. *If $V_1 \subset U_1$ then $U_2 \subset V_2$. Similarly with the indices swapped. Thus $V_i \subset U_i$ for both $i = 1, 2$ implies $V_i = U_i$ for both $i = 1, 2$.*

Proof. Assuming $V_1 \subset U_1$ we have $V_2^c \subset V_1 \subset U_1$ by definitions. Therefore, if $\exists x \in V_2^c \cap U_2$ then $x \in U_1 \cap U_2$. But this is a contradiction, if φ_1 and φ_2 are isomorphisms at x then $\mathcal{E}_i$'s are isomorphic at x. On the other hand $x \in V_2^c$ implies that the roots have different ranks at x. □

The following proves that $\mathcal{A}_1 \to \mathcal{A}$ is an open immersion.

Proposition 5.34. *Take a map $\operatorname{Spec} R \to \mathcal{A}$. Let $\mathfrak{p} \in \operatorname{Spec} R$ be a point such that $U_i(k) = V_i(k)$ for $i = 1, 2$ where k is the residue field of $\mathfrak{p}$. Then there exists a Zariski open neighbourhood W of $\mathfrak{p}$ such that $U_i(W) = V_i(W)$ for $i = 1, 2$.*

Proof. Let $j : C|_\mathfrak{p} \hookrightarrow C$ be the inclusion of the fiber over $\mathfrak{p}$. We will refer to $C|_\mathfrak{p}$ as the central fiber even though R is not necessarily local.

The fact that $U_i(k) = U_i(R)|_{\operatorname{Spec} k}$ implies that $U_i(R)$ is an open neighbourhood of of $j(U_i(k))$, similarly for V_i's. We know $V_i(k)$'s cover the central fiber and, by hypothesis, $U_i(k) = V_i(k)$. Therefore the open sets $U_i(R) \cap V_i(R)$ for $i = 1, 2$ cover the central fiber in C.

Pick a Zariski neighbourhood $W \subset \operatorname{Spec} R$ of $\mathfrak{p}$ such that the preimage of W is covered by $U_i(R) \cap V_i(R)$. Shrink W so that every component of the discriminant locus intersects the fiber over $\mathfrak{p}$. Let Z be the components of the discriminant locus on which the $\mathcal{E}_i$'s are both non-free. On $Z|_\mathfrak{p}$ the φ_i's are isomorphisms, hence they will remain an isomorphism in a neighbourhood of $Z \cap C_\mathfrak{p} \subset C$. Shrink W one last time so that the φ_i's are isomorphisms on all of Z.

Now we claim that $U_i(W) = V_i(W)$. By Lemma 5.33 it will be sufficient to show $V_i(W) \subset U_i(W)$ for $i = 1, 2$.

Pick $x \in V_1(W)$. We want to show $x \in U_1(W)$. If $x \in U_1(W) \cap V_1(W)$ then we are done. Otherwise, $x \in U_2(W) \cap V_2(W)$. Note $x \in V_1(W) \cap V_2(W)$ implies that either both $\mathcal{E}_i$'s are free or both $\mathcal{E}_i$ are non-free at x. Furthermore, $x \in U_2(W)$ implies φ_2 is an isomorphism at x.

If both the $\mathcal{E}_i$'s are free then the fact that φ_i's commute with b_i's imply that φ_1 is also an isomorphism. Hence $x \in U_1(W)$.

If the $\mathcal{E}_i$'s are both non-free, then $x \in Z$. But, by our construction of W, $x \in Z$ implies that φ_1 is an isomorphism at x. $\square$

5.3.2 Sync condition is closed

We now want to prove the following proposition. Its proof lasts until the end of Lemma 5.41.

Proposition 5.35. *The map $\mathcal{A}_2 \to \mathcal{A}_1$ is a closed immersion.*

Pick a morphism from a scheme $T \to \mathcal{A}_1$. Suppose a point $t \in T$ is such that $t \to \mathcal{A}_1$ factors through $\mathcal{A}_2 \to \mathcal{A}_1$. It will suffice to show $\mathcal{A}_2 \times_{\mathcal{A}_1} T \to T$ is a closed immersion in an étale neighbourhood of t.

We will rely heavily on formal neighbourhoods near the discriminant locus. To be able to transfer information from formal neighbourhoods we need to be able to assume that T is locally noetherian. This is possible if $\mathcal{A}_1$ is locally noetherian.

Lemma 5.36. *$\mathcal{A}_1 \to S$ is locally of finite type. In particular, $\mathcal{A}_1$ is locally noetherian.*

Proof. We have the following chain of maps:

$$\mathcal{A}_1 \to \mathcal{X} \to \overline{\mathcal{S}}^{m-1}(\mathcal{N}) \times_\mathcal{M} \overline{\mathcal{S}}(\mathcal{N}) \to \mathcal{M} \to S.$$

The map $\mathcal{A}_2 \to \mathcal{X}$ is an open immersion as proved in Proposition 5.34. The map $\mathcal{Y} := \overline{\mathcal{S}}^{m-1}(\mathcal{N}) \times_\mathcal{M} \overline{\mathcal{S}}^{m-1}(\mathcal{N}) \to S$ is locally of finite type, as part of our induction hypothesis, and the map $\mathcal{X} \to \mathcal{Y}$ is locally of finite type as shown in [Hal]. Thus the composition $\mathcal{A}_1 \to S$ is locally of finite type.

Since S is excellent, it is noetherian. Thus $\mathcal{A}_1$ is locally noetherian. □

Using the lemma above and by passing to an étale neighbourhood of t, we may assume $T = \operatorname{Spec} R$ is noetherian affine and that the discriminant locus of the curve $C_T \to T$ has n components $Z_1, \dots, Z_n$ such that $Z_i \cap C_t = v_i$ where $v_i \in C_t$ is a single node of C_t.

As we mentioned in Remark 5.31 we may assume $m = 2$ whenever our constructions are local in C. This being the case for the rest of this section, from now on we will assume $m = 2$.

Denote by $(\mathcal{E}_i, b_i)_{i=1}^2$ the two roots on C_T and let $(\tau_i : \mathcal{F} \to \mathcal{E}_i^2)_{i=1}^2$ be the pre-sync data associated to the map $T \to \mathcal{A}_1$.

Definition 5.37. Let $F_i \subset \hom(\_, (T,t))$ be the subfunctor defined so that $(T', t') \to (T, t)$ belongs to F_i iff $(\tau_1, \tau_2)|_{T'}$ gives a sync data around $Z_i|_{T'}$.

We only need to check the sync condition in a neighbourhood of each Z_i. For this reason, the morphism $\mathcal{A}_2 \times_{\mathcal{A}_1} T \to T$ can be viewed as a fiber product $F_1 \times_T \cdots \times_T F_n \to T$.

We have thus reduced our goal to proving that $F_i \to T$ is a closed immersion for all i. If one of the two roots is free around Z_i then this problem is trivial since $F_i = \hom(\_, T)$.

So, fix a node $v_i \in C_t$ where both roots are non-free. At this point, we may assume that there is a single node $v = v_i \in C_T$ and the discriminant locus $Z = Z_i$ is connected. Let F stand for F_i.

In a Zariski neighbourhood V of Z both the morphisms τ_1 and τ_2 are isomorphisms, hence we can define $\psi := \tau_2 \circ \tau_1^{-1}$ on V. Our goal can then be roughly described as finding the conditions on the base T for which ψ induces an isomorphism $\operatorname{Sym}^{2*} \mathcal{E}_1|V \xrightarrow{\sim} \operatorname{Sym}^{2*} \mathcal{E}_2|_V$.

By passing to an étale neighbourhood of t in T if necessary, we may assume the existence of an étale neighbourhood $U \to C_T$ of v satisfying the following three conditions:

- U is affine, i.e., $U = \operatorname{Spec} A$.
- The image of $U \to C_T$ contains Z.
- For each $i = 1, 2$ we have $(\mathcal{E}_i, b_i)|_U \simeq (E(p_i, p_i), s)$ for some $p_i \in R$.

Here $E(p_i, p_i)$ is defined as in Section 2.4.1. For the possibility of assuming the third condition, see Theorem 3.9 [Fal96].

Definition 5.38. An étale neighbourhood $U \to C_T$ of Z satisfying the three conditions above will be called a *preferred neighbourhood* of Z.

We also fix $x, y \in A$ satisfying the following conditions:

- The discriminant locus Z_U of $U \to T$ is the closed subscheme corresponding to (x, y).

- $xy = \pi \in R$.
- The corresponding map $R[x, y]/(xy - \pi) \to A$ induces an isomorphism when both sides are completed with respect to the ideal (x, y).

Remark 5.39. The local deformation theory of stable curves guarantees the existence of such x, y possibly after some étale base change. For instance, it follows immediately from [Stacks, Tag 0CBY].

Remark 5.40. Jarvis [Jar98] calls such $x, y \in A$ *local coordinates.*

Now we are ready to prove the main lemma of this section.

Lemma 5.41. *$F \to T$ is a closed immersion.*

Proof. We pick local coordinates $x, y \in A$ around a preferred neighbourhood $U = \operatorname{Spec} A \to C$ around the discriminant locus.

Denote by Z the discriminant locus of $U \to T$ which is carved out by $J = (x, y)$ and $xy = \pi \in R$. By definition, completion of A with respect to J satisfies $\hat{A}_J \xrightarrow{\sim} \hat{R}_{(\pi)}[[x, y]]/(xy - \pi)$. Let $\hat{Z} := \operatorname{Spec} \hat{A}_J$ and note that the map $(U \setminus Z) \sqcup \hat{Z} \to U$ is an fpqc cover.

Let $\psi : (E(p_1, p_1)^2, s) \to (E(p_2, p_2)^2, s)$ be an isomorphism induced by the pre-sync data. Consider the morphism $\psi' : \operatorname{Sym}^2 \mathcal{E}_1^2 \to \mathcal{E}_2^4$ induced from ψ. Denote the standard generators by $\xi_1, \xi_2 \in E(p_1, p_1)$ and $\zeta_1, \zeta_2 \in E(p_2, p_2)$.

Recall from Lemma 5.22 that ψ is a sync data iff $\psi'(\xi_1^2 \xi_2^2 - (\xi_1 \xi_2)^2) = 0$. This is always satisfied on $U \setminus Z$, so we concentrate on $\hat{Z}$.

Using standard arguments (§A2.3 [Eis95]) one concludes that:

$$E(p, p)^2 = \operatorname{coker}\left(L(p, p) : \hat{A}_J^{\oplus 4} \to \hat{A}_J^{\oplus 3} \right)$$

where

$$L(p, p) = \begin{pmatrix} y & -p & 0 & 0 \\ -p & x & y & -p \\ 0 & 0 & -p & x \end{pmatrix}. \tag{5.3.1}$$

We want to lift ψ to a map $\hat{A}_J^{\oplus 3} \to \hat{A}_J^{\oplus 3}$. Using the 4 relations above we construct a lift such that the corresponding 3×3 matrix contains no terms involving y in the first row, x or y in the second row, x in the third row.

Such a lift is unique and will be denoted by $[\psi]$. A direct calculation shows that ψ commutes with the two s's iff the lift $[\psi]$ satisfies the following equality

$$[\psi] = \begin{pmatrix} 1 & 0 & 0 \\ a_1 & u & a_2 \\ 0 & 0 & 1 \end{pmatrix} \tag{5.3.2}$$

where $a_1, a_2 \in \operatorname{Ann}(p_2) = \operatorname{Ann}(p_1)$ and $u \in R^\times$ is such that $up_2 = p_1$.

In terms of the entries of this matrix we have the following equality:

$$\psi'(\xi_1^2\xi_2^2 - (\xi_1\xi_2)^2) = \zeta_1^2\zeta_2^2 + a_1\zeta_2^2(\zeta_1\zeta_2) + a_2\zeta_1^2(\zeta_1\zeta_2) + (a_1a_2 - u^2)(\zeta_1\zeta_2)^2.$$

As before we can calculate a presentation of $E(p_2, p_2)^4$. This presentation looks similar to Equation 5.3.1 but with an extra block. This shows that ψ is a sync data iff $a_1 = a_2 = 0$ and $u^2 = 1$. Since we assume $\operatorname{Spec} R$ is connected, this forces $u = \pm 1$.

In fact, u can attain only one of these values. To determine which, we can pullback our representation $[\psi]$ to t. By the previous paragraph, and our hypothesis on t, there is an $\varepsilon \in \{\pm 1\}$ such that:

$$[\psi_t] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Define an ideal $I \subset R$ so that $\hat{I}_{(\pi)} \subset \hat{R}_{(\pi)}$ equals $(a_1, u - \varepsilon, a_2)$ and $I|_{D(\pi)} = 0$ on the principal open subset $D(\pi) \subset T$.

Along any $T' \to T$ let Z' be the discriminant locus of $U' = U \times_T T' \to T'$. This gives us a commutative map:

$$\begin{array}{ccc} \hat{Z}' & \longrightarrow & \hat{Z} \\ \downarrow & & \downarrow \\ U' & \longrightarrow & U \end{array}$$

The matrix entries of $[\psi]$ pullback as expected from Z to Z'. Therefore $\psi_{U'}$ satisfies the sync condition iff the map $T' \to T$ sends the three terms $a_1, a_2, u - \varepsilon$ to 0. Therefore, F is represented by $\operatorname{Spec} R/I$. $\square$

At this point we proved Proposition 5.35. However, we make one final deduction from the contents of this section.

Set-up 5.42. Assume R to be a complete local noetherian ring with an algebraically closed residue field.

Lemma 5.43. *With R as in Set-up 5.42 and $C \to \operatorname{Spec} R$ a stable curve, the sync condition can be checked in the formal neighbourhood of the discriminant locus.*

Proof. This follows from the proof of the previous result, with the additional observation that no étale base change is required. $\square$

Corollary 5.44. *When the base ring is as in Set-up 5.42, the two definitions of a multiple root given by Definition 3.4 and Definition 5.21 agree.*

Proof. Use the lemma above to reduce the problem to the formal neighbourhood of the discriminant locus. Lemma 2.22 describes the isomorphisms explicitly that are allowed by Definition 3.4. On the other hand, in the proof of Lemma 5.41 we gave explicit conditions for the sync condition. The two descriptions agree. $\square$

5.3.3 The diagonal

Remark 5.45. Now that we proved in Theorem 5.29 that $\overline{\mathcal{S}}^m(\mathcal{N})$ is algebraic, it follows that the relative diagonal $\overline{\mathcal{S}}^m(\mathcal{N}) \to \mathcal{M}$ is representable by algebraic spaces.

Proposition 5.46. *The diagonal* $\Delta : \overline{\mathcal{S}}^m(\mathcal{N}) \to \overline{\mathcal{S}}^m(\mathcal{N}) \times_{\mathcal{M}} \overline{\mathcal{S}}^m(\mathcal{N})$ *is finite and unramified.*

Proof. Fix a morphism $B \to \overline{\mathcal{S}}^m(\mathcal{N}) \times_{\mathcal{M}} \overline{\mathcal{S}}^m(\mathcal{N})$ where B is a scheme. Since the stacks under consideration are locally noetherian and the question is local on the target of Δ we may assume B to be noetherian.

This morphism defines a curve over B and a pair of multiple roots $\mathfrak{R} = (\mathcal{R}, \Phi)$ and $\mathfrak{R}' = (\mathcal{R}', \Phi')$. We want to show that the functor $\mathrm{Iso}(\mathfrak{R}, \mathfrak{R}')$ is represented by a finite and unramified scheme over B. In light of Remark 5.45 the map Δ is representable by algebraic spaces.

An isomorphism of a multiple root is an isomorphism of the underlying sequence of roots compatible with the sync data. Therefore, we have a map $\mathrm{Iso}(\mathfrak{R}, \mathfrak{R}') \to \mathrm{Iso}(\mathcal{R}, \mathcal{R}') = \prod_{i=1}^m \mathrm{Iso}(\mathcal{R}_1, \mathcal{R}'_1)$, where the product of the functors $\mathrm{Iso}(\mathcal{R}_i, \mathcal{R}'_i)$ is to be taken over B.

Each of the isomorphism functors $\mathrm{Iso}(\mathcal{R}_i, \mathcal{R}'_i)$ is represented by a finite and unramified scheme over B as shown in §4.1.4.3 of [Jar98]. Hence, their product over B is also finite and unramified over B. We will now show that $\mathrm{Iso}(\mathfrak{R}, \mathfrak{R}')$ is a connected component of $\mathrm{Iso}(\mathcal{R}, \mathcal{R}')$, and hence finite and unramified over B.

To show that $\mathrm{Iso}(\mathfrak{R}, \mathfrak{R}')$ is a component of $\mathrm{Iso}(\mathcal{R}, \mathcal{R}')$, we will prove that the property of being compatible with the sync data for a sequence of isomorphisms both specializes and generalizes. Since we now know that the diagonal is representable this will conclude the proof. Furthermore, since $\mathrm{Iso}(\mathcal{R}, \mathcal{R}')$ is locally noetherian we need only use *discrete* valuation rings.

Let R be a complete DVR with generic point η and special point σ. Consider a map $\operatorname{Spec} R \to \mathrm{Iso}(\mathcal{R}, \mathcal{R}')$, which gives us a family of curves $\mathcal{X} \to \operatorname{Spec} R$ and a sequence of isomorphism $(\varphi_i : (\mathcal{E}_i, b_i) \to (\mathcal{E}'_i, b'_i))_{i=1}^m$ between the roots.

We can focus on one node at a time because distinct nodes do not interact with one another. In light of Lemma 5.44, we can use the simpler definition of a multiple root defined for complete noetherian local rings. But with this definition, compatibility with the sync data has to be checked for each pair of indices i, j separately. Therefore, we can assume $m = 2$ to simplify notation.

Pick a component of the discriminant locus. We need only focus our attention to the formal completion of this locus. Recall that in the formal neighbourhood of a node the isomorphisms φ_i between $\mathcal{E}_i$ and $\mathcal{E}'_i$ are of the form $\begin{bmatrix} \varepsilon'_i & 0 \\ 0 & \varepsilon''_i \end{bmatrix}$, with $\varepsilon'_i, \varepsilon''_i \in \{\pm 1\}$ (this is Lemma 2.22). The symmetric square φ_i^2 equals $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \varepsilon_i & 0 \\ 0 & 0 & 1 \end{bmatrix}$, where $\varepsilon_i = \varepsilon'_i \varepsilon''_i$.

In the meantime, the sync condition on either side is represented by an isomorphism $\psi : \mathcal{E}_1^2 \to \mathcal{E}_2^2$ on the first pair of roots and ψ' on the second pair of roots. We know that $\psi = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $\psi' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \varepsilon' & 0 \\ 0 & 0 & 1 \end{bmatrix}$ where $\varepsilon, \varepsilon' \in \{\pm 1\}$.

These four isomorphisms give a commuting diagram iff $\varepsilon \varepsilon_1 = \varepsilon' \varepsilon_2$. Therefore we have reduced the argument to showing that this equality holds over the special fiber iff it holds over the generic fiber.

First and foremost, if the component of the discriminant locus we are studying is an isolated node, then $\pi \in R$ representing this node is not zero. In particular, this forces that there be precisely one isomorphism between the squares of these roots (see Remark 2.24). Therefore the desired equality holds by uniqueness.

On the other hand, if the node is defined over the generic fiber then it persists over the entire base. In this case, each of the signs ε_i are constant throughout this discriminant locus. Therefore, the equality $\varepsilon\varepsilon_1 = \varepsilon'\varepsilon_2$ holds iff it holds at any one part of this locus. □

Corollary 5.47. *$\overline{\mathcal{S}}^m(\mathcal{N})$ is a Deligne–Mumford stack.*

Proof. The proposition above combined with the fact that $\mathcal{M}$ is a DM stack yields this result. □

5.4 Additional properties

5.4.1 Smoothness

For the first half of Part I we studied the local structure of our stacks $\overline{\mathcal{S}}^m(\mathcal{N})$. The following is a useful summary of what we know.

Theorem 5.48. *If $\mathcal{M} \to \overline{\mathcal{M}}_g$ is smooth then $\overline{\mathcal{S}}^m(\mathcal{N}) \to S$ is smooth.*

Proof. Smoothness can be checked in geometric formal neighbourhoods of points. But smoothness at geometric formal neighbourhoods is precisely the content of Corollary 4.23. □

Remark 5.49. Conceptually, we are not just proving that $\overline{\mathcal{S}}^m(\mathcal{N})$ is smooth but also that our definition of multiple roots is the "right" one, since we can now use intersection theory on the stacks $\overline{\mathcal{S}}^m(\mathcal{N})$ to solve enumerative problems.

5.4.2 A proper compactification

We now prove that $\overline{\mathcal{S}}^m(\mathcal{N})$ is indeed a 'closure' of $\mathcal{S}^m(\mathcal{N})$ when we would expect.

Lemma 5.50. *If $\mathcal{C} \to \mathcal{M}$ is generically smooth then $\mathcal{S}^m(\mathcal{N}) \to \overline{\mathcal{S}}^m(\mathcal{N})$ is a dense open immersion.*

Proof. Since all roots on a smooth curve are locally free, it suffices to show that any root can be deformed onto a smooth curve. Provided that any singular curve X can be deformed to a smooth curve over $\mathcal{M}$, it follows immediately from the local deformation functors discussed in Chapter 4 that any tuple of roots on X can also be deformed onto a smooth curve over $\mathcal{M}$. In particular, this is immediate from Theorem 4.21. □

Remark 5.51. If $\mathcal{C} \to \mathcal{M}$ is not assumed to be generically smooth the result will certainly not hold, even when $m = 1$. For example one could take $\mathcal{M} = \operatorname{Spec} k$. In this case, the isomorphism classes of roots of a fixed line bundle form a discrete set so one can not deform the locally free roots on to the non-free roots.

Lemma 5.52. *The morphism $\overline{\mathcal{S}}^m(\mathcal{N}) \to \mathcal{M}$ is proper.*

Proof. We prove this by the valuative criterion of properness and induction on $m \geq 1$. The result for $m = 1$ is part of Proposition 5.3. Then we need only show that the map $\mathcal{Y} := \overline{\mathcal{S}}^m(\mathcal{N}) \to \overline{\mathcal{S}}^{m-1}(\mathcal{N}) \times_{\mathcal{M}} \overline{\mathcal{S}}(\mathcal{N})$ defined in Remark 5.27 is proper. Clearly the diagonal is locally noetherian so we may restrict to checking the valuative criterion using complete DVRs.

Let R be a complete discrete valuation ring, with residue field K. Consider a 2-commutative diagram:

$$\begin{tikzcd}
\operatorname{Spec} K \arrow[r] \arrow[d] & \overline{\mathcal{S}}^m(\mathcal{N}) \arrow[d] \\
\operatorname{Spec} R \arrow[r] & \mathcal{Y}
\end{tikzcd}$$

This means that we have a synchronized $(m-1)$-tuple of roots and an m-th root over the curve $C_R \to \operatorname{Spec} R$. Furthermore, these roots are all synchronized over the general fiber. However, as we demonstrated in the proof of Proposition 5.46 a synchronization on the generic fiber over a complete DVR extends to the entire family uniquely. $\square$

5.4.3 Coarse moduli space

Lemma 5.53. $\overline{\mathcal{S}}^m(\mathcal{N}) \to \mathcal{M}$ *is quasi-finite.*

Proof. We will use the fact that $\overline{\mathcal{S}}(\mathcal{N}) \to \mathcal{M}$ is quasi-finite, see Proposition 5.9. Fix a geometric point of the m-fold product $\overline{\mathcal{S}}^m(\mathcal{N}) \times_{\mathcal{M}} \cdots \times_{\mathcal{M}} \overline{\mathcal{S}}(\mathcal{N})$. Our goal is to show that there are finitely many synchronizations on the corresponding sequence of roots.

In light of Corollary 5.44 we may use Definition 3.4. But then we may reduce to the case $m = 2$. In the formal neighbourhood of each node where both roots are singular, we need to show that the number of isomorphism between the two roots is finite. This is implied by Lemma 2.22 which says that there are precisely 4 such isomorphisms. $\square$

Remark 5.54. The proof below is a slight adaptation of Proposition 3.1.1 [Jar00].

Proposition 5.55. *If the coarse moduli space of $\mathcal{M}$ is projective over S then the coarse moduli space of $\overline{\mathcal{S}}^m(\mathcal{N})$ is projective over S.*

Proof. It is well known that separated Deligne–Mumford stacks are coarsely represented by algebraic spaces (e.g. Corollary 1.3.1 [KM97]). So we let $X = \mathfrak{coarse}(\overline{\mathcal{S}}^m(\mathcal{N}))$ and $Y = \mathfrak{coarse}(\mathcal{M})$ be these coarse moduli spaces with $f : X \to Y$ the natural map between them.

This map f is proper because the corresponding map between the stacks is proper. Also f is quasi-finite by Lemma 5.53. Therefore f is finite and hence projective. When $Y \to S$ is projective then so is $X \xrightarrow{f} Y \to S$. $\square$

5.5 Non-normal product space

We claimed in the introduction that the m-fold product $\overline{\mathcal{S}}(\mathcal{N})\times_{\mathcal{M}}\cdots\times_{\mathcal{M}}\overline{\mathcal{S}}(\mathcal{N})$ is non-normal in general. We will prove this here.

For the purposes of demonstration, it will be sufficient to assume $m = 2$, $\mathcal{M} = \overline{\mathcal{M}}_g$ the moduli space of stable curves of genus g with $\mathcal{C} \to \mathcal{M}$ the universal curve over it. Take $\mathcal{N} = \omega_{\mathcal{C}/\mathcal{M}}$ to be the relative dualizing sheaf.

Let $\mathcal{Y} = \overline{\mathcal{S}}(\mathcal{N}) \times_{\mathcal{M}} \overline{\mathcal{S}}(\mathcal{N})$. For k an algebraically closed field, pick a general irreducible curve C over k having a single node $x \in C$. This corresponds to a point $c : \operatorname{Spec} k \to \mathcal{M}$. Let $(\mathcal{E}_i, b_i)$ for $i = 1, 2$ be non-isomorphic roots of $\omega_{C/k}$ such that the $\mathcal{E}_i$'s are both non-free at the node x. This corresponds to a point $y \to \mathcal{Y}$ lying over c.

Proposition 5.56. *The moduli stack $\mathcal{Y}$ is non-normal at y.*

Proof. Using Theorem 4.21 for each of the roots seperately, and arguing as in Section 3.2 we conclude that the local deformation functor at y of $\mathcal{Y}$ is pro-represented by

$$\Lambda[[\tau]]] \times_{\Lambda[[\tau^2]]} \Lambda[[\tau]] \simeq \Lambda[[\tau_1, \tau_2]]/(\tau_1^2 - \tau_2^2),$$

where Λ is smooth over the base scheme S.

Therefore $\mathcal{Y}$ is non-normal at y. However, it may still be that the coarse moduli space of $\mathcal{Y}$ is normal at the image of y. □

In passing to coarse moduli spaces, we need to compute the autormophism groups of the spin curves and their action on the functor of deformations. Since the generic singular curve without elliptic tails will have no inessential automorphisms, our work in Section 4.5 gives the action of the full automorphism group. In particular, $\operatorname{Aut}(y) = \{\pm 1\}$ and the action of $\operatorname{Aut}(y)$ is trivial on Def_y.

This concludes our argument that even on the coarse moduli space the image of y is contained in a non-normal locus. Hence we proved the following. The second part is immediate as the coarse moduli of a smooth Deligne–Mumford stack is always normal.

Proposition 5.57. *The coarse moduli space of $\overline{\mathcal{S}}(\mathcal{N}) \times_{\mathcal{M}} \overline{\mathcal{S}}(\mathcal{N})$ is non-normal, whereas the coarse moduli space of $\overline{\mathcal{S}}^2(\mathcal{N})$ is normal.*

Part II

Double spin curves and the double contact loci

Chapter 1

Boundary divisors

The first part of this thesis gave a compactification $\overline{\mathcal{S}}_g^{--}$ of the moduli space $\mathcal{S}_g^{--}$ of double odd spin curves. In this chapter we will study $\overline{\mathcal{S}}_g^{--}$ in greater detail, concentrating on the boundary $\overline{\mathcal{S}}_g^{--} \setminus \mathcal{S}_g^{--}$.

Definition 1.1. If $\mathcal{M}$ is a moduli space and $\overline{\mathcal{M}}$ is a compactification of $\mathcal{M}$ then the difference $\overline{\mathcal{M}} \setminus \mathcal{M}$ is often referred to as the *boundary of* $\overline{\mathcal{M}}$.

The emphasis on Part I was on families of curves. In this second part, individual curves play a more central role. Moreover, we will almost exclusively be interested in roots of the canonical bundle. For these reasons we will begin anew by explaining individual spin and double spin curves in detail, unpacking the constructions of the first part.

Although we will concentrate only on $\overline{\mathcal{S}}_g^{--}$, the arguments here apply with minimal modification to the boundary of $\overline{\mathcal{S}}_g^{-+}, \overline{\mathcal{S}}_g^{+-}, \overline{\mathcal{S}}_g^{++}$.

1.1 Boundary of the moduli of stable curves

In order to setup notation we will recall the general properties of the moduli space of stable curves, $\overline{\mathcal{M}}_g$. It is well known that the boundary $\Delta := \overline{\mathcal{M}}_g \setminus \mathcal{M}_g$ consists of $\lfloor \frac{g}{2} \rfloor$ irreducible components: $\Delta = \Delta_0 \cup \Delta_1 \cup \cdots \cup \Delta_{\lfloor \frac{g}{2} \rfloor}$.

Definition 1.2. Let $X \in \Delta \subset \overline{\mathcal{M}}_g$ be a stable curve over $\operatorname{Spec} k$. If the partial normalization $\tilde{X}$ of X at a node $x \in X$ is disconnected then x is called a *disconnecting node*. In this case, if $\tilde{X} = C_1 \sqcup C_2$ then letting $i = \min(g(C_1), g(C_2))$ we will call x a *node of type* i. The curve X will also be said to be of type i. If x is not a disconnecting node we will cal it a *node of type 0*. This terminology is taken from [DM69].

Then for each $i = 0, \ldots, \lfloor \frac{g}{2} \rfloor$ the closed points of the boundary component Δ_i parametrizes the stable curves of type i. More specifically, Δ_i is the image of the clutching morphism $\overline{\mathcal{M}}_{i,1} \times \overline{\mathcal{M}}_{g-i,1} \to \overline{\mathcal{M}}_g$ for $i > 0$ and of $\overline{\mathcal{M}}_{g-1,2} \to \overline{\mathcal{M}}_g$ when $i = 0$.

1.2 Degenerate spin curves

Here we will briefly recall the definition of a spin structure on a singular, but stable, curve. The notion has been introduced by Cornalba [Cor89], generalized by [Jar98] using torsion-free sheaves and by [CCC07] using quasi-stable curves. We describe in Chapter III.2 the equivalence between the two approaches. As both approaches have their advantages, we will go back and forth between the two according to our needs.

Most of the work in Part II relies on individual curves over k as opposed to families of curves. Therefore, we will define degenerate spin curves over k and illustrate the concept with examples which we will later use. However, for the definition of a family of spin curves in general, we refer to Part I. In general we will omit any proofs during this section, implicitly referring to the main references or to Part I.

1.2.1 Stable spin curves

Let $\overline{X}$ be a stable curve and $\mathcal{E}$ a coherent, torsion-free $\mathcal{O}_{\overline{X}}$-module of rank-1. Suppose $\alpha : \mathcal{E}^{\otimes 2} \to \omega_{\overline{X}}$ is a non-degenerate map, i.e., the induced map $\mathcal{E} \to \omega_{\overline{X}} \otimes \mathcal{E}^{\vee}$ is an isomorphism.

Definition 1.3. The triplet $(\overline{X}, \mathcal{E}, \alpha)$ is called a *stable spin curve*. The pair $(\mathcal{E}, \alpha)$, and often just $\mathcal{E}$, will be called a *stable spin structure* or a *stable theta characteristic* on $\overline{X}$.

Remark 1.4. As a line bundle is torsion-free and of rank-1, the usual notion of a square root of the canonical bundle is included in this definition. Moreover, if $\overline{X}$ is smooth then all rank-1 torsion-free sheaves are line bundles.

Definition 1.5. If $(\overline{X}, \mathcal{E}, \alpha)$ is a stable spin curve and $\mathcal{E}$ is not locally free at a node $x \in \overline{X}$ then x will be called a *singularity of* $\mathcal{E}$, or a *singularity of the spin structure*. Otherwise, $\mathcal{E}$ will be said to be smooth at x. If $x \in \overline{X}$ is a node of type i and $\mathcal{E}$ is singular at x then we will refer to x as a *type i singularity* of the spin curve. In this case, the spin curve will be called *of type i*.

1.2.2 Quasi-stable spin curves

Definition 1.6. Let X be a connected nodal curve such that each unstable component $E \subset X$ is of genus 0 and such that E contains 2 nodes of X. If, furthermore, no two unstable components of X intersect then X is called a *quasi-stable curve*. The unstable components of X are called *exceptional components of X*.

Definition 1.7. Let X be a quasi-stable curve of genus g. Let L be a degree $g-1$ line bundle on X such that L is of degree 1 on each exceptional component. Suppose $\beta : L^{\otimes 2} \to \omega_X$ is a morphism which is an isomorphism in the complement of the exceptional components. Then (X, L, β) is called a *quasi-stable spin curve*.

Lemma 1.8. *There is a natural correspondence between stable and quasi-stable spin curves.*

Proof. Let $(\overline{X}, \mathcal{E}, \alpha)$ be a stable spin curve and let $X = \underline{\mathrm{Proj}}(\mathrm{Sym}^* \mathcal{E})$ with $\mathcal{O}_X(1)$ the line bundle coming from the projectivization. There is a natural map $\beta : \mathcal{O}_X(2) = \mathcal{O}_X(1)^{\otimes 2} \to \omega_X$ induced by α. The tuple $(X, \mathcal{O}_X(1), \beta)$ is a quasi-stable spin curve. If we pushforward $(\mathcal{O}_X(1), \beta)$ we recover $(\mathcal{E}, \alpha)$. See Chapter III.2 for more details. □

Let $(\overline{X}, \mathcal{E}, \alpha)$ be a stable spin curve with $x \in \overline{X}$ a singularity of type i of the spin structure, in particular, $\dim_k \mathcal{E}|_x = 2$. Let $\nu : \tilde{X} \to \overline{X}$ be the partial normalization of $\overline{X}$ at x.

Lemma 1.9. *There is a coherent sheaf $\tilde{\mathcal{E}}$ on $\tilde{X}$, restricting to a stable spin structure on each* connected *component of $\tilde{X}$ such that $\mathcal{E} \simeq \nu_* \tilde{\mathcal{E}}$*

Proof. See Construction 4.1.2 of [Jar98]. □

This observation allows us to classify all spin structures on $\overline{X}$ by induction. Note that, not every spin structure on $\overline{X}$ needs to have a singular point. However, if $\mathcal{E}$ has no singular points then all nodes of $\overline{X}$ are of type 0.

For the purposes of this work we will only be interested in $\overline{X}$ having a single node. Here are the three archetypical examples that we will work with. We provide both the stable and the quasi-stable models for each example, since we will change back and forth between the two interpretations.

1.3 Degenerate spin curves

Here we will give examples of spin curves that are general in the boundary of $\overline{\mathcal{S}}_g^-$. We will first do this in detail for the stable models, then explain how to pass to the quasi-stable model and list the results of this construction. We systematically ignore the squaring maps for simplicity of exposition.

Example 1.10 (Generic curve in Δ_0^b)**.** Let $\overline{X}$ be a type 0 stable curve of genus g with a single node x. Let $\nu : C \to \overline{X}$ be the normalization map. Pick a theta characteristic η on C and let $\mathcal{E} = \nu_* \eta$. Then $\mathcal{E}$ is a stable spin structure on $\overline{X}$. Note that $h^0(\mathcal{E}) = h^0(\eta)$ so the parity of $\mathcal{E}$ and of η agree.

Example 1.11 (Generic curve in Δ_0^n)**.** Let $\overline{X}$ and $\nu : C \to \overline{X}$ be as in Example 1.10. Then the line bundle $\omega_{\overline{X}}$ admits square roots that are locally free. Any such root pulls back on C to the root of $\omega_C(p+q)$ where $p, q \in C$ are the preimages of the node $x \in \overline{X}$. Conversely, if $\tau \in \sqrt{\omega_C(p+q)}$ is a line bundle then there is precisely one isomorphism $h : \tau|_p \xrightarrow{\sim} \tau|_q$ such that the line bundle $L = (\tau, h)$ on X obtained by gluing the fibers of τ using h satisfies $h^0(C, \tau) = h^0(X, L)$. The line bundle L is an odd theta characteristic (see Corollary 1.20), to get an even theta characteristic one would use the isomorphism $-h$ instead.

Example 1.12 (Generic curve in Δ_i^+)**.** Let $\overline{X}$ be a stable curve of genus g with a single node x. Suppose x is a node of type $i > 0$. Let C_1, C_2 be the two components of $\overline{X}$ and $\iota_i : C_i \hookrightarrow \overline{X}$ their inclusion maps. If η_i is a theta characteristic on C_i then $\mathcal{E} := \iota_{1,*}\eta_1 \oplus \iota_{2,*}\eta_2$

is a (stable) spin structure on $\overline{X}$. Furthermore, $h^0(\overline{X}, \mathcal{E}) = h^0(C_1, \eta_1) + h^0(C_2, \eta_2)$ so the parity of $\mathcal{E}$ is the sum of parities of η_1 and η_2. In particular, $\mathcal{E}$ is odd iff η_1 and η_2 have opposite parities. By Lemma 1.9, all spin structures on $\overline{X}$ are obtained in this way.

Using the construction in the proof of Lemma 1.8 we pass to the quasi-stable models, and record here the results of this operation on the examples given above. We will continue to use the notation introduced in the corresponding examples.

Example 1.13 (Generic curve in Δ_0^b)**.** We have $X = C \cup_{\substack{p\sim 0\\ q\sim\infty}} \mathbb{P}^1$ and $L|_C \simeq \eta$, $L|_{\mathbb{P}^1} \simeq \mathcal{O}_{\mathbb{P}^1}(1)$.

Example 1.14 (Generic curve in Δ_0^n)**.** We have $X = \overline{X}$ and $L|_C \simeq \tau \in \sqrt{\omega_C(p+q)}$.

Example 1.15 (Generic curve in Δ_i^+)**.** We have $X = C_1 \cup_{p\sim 0} \mathbb{P}^1 \cup_{\infty\sim q} C_2$ and $L|_{C_i} \simeq \eta_i$, $L|_{\mathbb{P}^1} \simeq \mathcal{O}_{\mathbb{P}^1}(1)$.

1.4 Clutching morphisms, I

For $i = 1, \ldots, g-1$ we mentioned the clutching maps $\overline{\mathcal{M}}_{i,1} \times \overline{\mathcal{M}}_{g-i,1} \to \overline{\mathcal{M}}_g$ whose images are denoted by Δ_i. For $i = 0$ we have $\overline{\mathcal{M}}_{g-1,2} \to \overline{\mathcal{M}}_g$ whose image is denoted by Δ_0. Since the domain of these maps are irreducible, their images are also irreducible. We will employ a similar argument for the moduli spaces of spin curves and double spin curves.

The boundary of the moduli space of spin curves has been discussed by Cornalba [Cor89]. However, as far as we are aware, the clutching maps between spin curves has not been made explicit. Therefore, we will do this here.

We take [Knu83] as basis for the clutching maps for families of curves and add onto it the clutching of the spin structures. The proofs of the lemmas below describe the clutching maps explicitly.

Lemma 1.16. *Let $i \in \{1, \ldots, g-1\}$. There is a clutching map $\overline{\mathcal{S}}^+_{i,1} \times \overline{\mathcal{S}}^-_{g-i,1} \to \overline{\mathcal{S}}^-_g$ whose image consists of all spin curves with a type i node.*

Proof. Let $B \to \overline{\mathcal{S}}^+_{i,1} \times \overline{\mathcal{S}}^-_{g-i,1}$ be a morphism from a scheme which then corresponds to a pair of families of stable spin curves $(\mathcal{X}_j \xrightarrow{\pi_j} B, \mathcal{E}_j, \alpha_j : \mathcal{E}_j^{\otimes 2} \to \omega_{\pi_j})$ together with a section $\sigma_j : B \to \mathcal{X}_j$. Here we take the first family, $j = 1$, to be the family of genus i. We can glue the underlying curves along the sections to get a family of curves $\pi : \mathcal{X} \to B$, two inclusions $\iota_j : \mathcal{X}_j \to \mathcal{X}$ and a section $\sigma = \iota_j \circ \sigma_j$, $j = 1, 2$.

Let $\mathcal{E} = \iota_{1,*}\mathcal{E}_1 \oplus \iota_{2,*}\mathcal{E}_2$. An easy way to see that this is flat is to realize that the construction can first be performed over the universal curves and then pulled back to B. But the relevant moduli spaces are reduced and $\mathcal{E}$ clearly has constant Hilbert polynomial on the fibers. There is a natural inclusion map $\iota_{1,*}\omega_{\pi_1} \oplus \iota_{2,*}\omega_{\pi_2} \hookrightarrow \omega_\pi$ and its composition with $\iota_{1,*} \circ \alpha_1 \oplus \iota_{2,*} \circ \alpha_2$ defines a map $\alpha : \mathcal{E}^{\otimes 2} \to \omega_\pi$. It is straight forward to check that $(\mathcal{E}, \alpha)$ is a spin structure on $\pi : \mathcal{X} \to B$ and that this construction is functorial. Therefore we get the desired clutching map.

It is clear that type i spin curves over k will lie in the image: just perform the gluing operation above with $B = \operatorname{Spec} k$. This implies that if there is a family of type i spin curves over some base T then, étale locally on T, the family is in the image of this clutching map. $\square$

Lemma 1.17. *There is a clutching map $\overline{\mathcal{S}}^{-}_{g-1,2} \to \overline{\mathcal{S}}^{-}_{g}$ whose image consists of spin curves of type* 0.

Proof. Let $(\mathcal{X} \xrightarrow{\pi} B, \mathcal{E}, \alpha : \mathcal{E}^{\otimes 2} \to \omega_\pi)$ with disjoint sections $\sigma_1, \sigma_2 : B \to \mathcal{X}$ correspond to some map $B \to \overline{\mathcal{S}}^{--}_{g-1,2}$. Glue the stable curve $\mathcal{X}$ along the two sections to get a family of stable curves $\nu : \mathcal{Y} \to B$, and pushforward $\mathcal{E}$ on to $\mathcal{Y}$. As in the proof of lemma above, we can define $\beta : \mathcal{E}^{\otimes 2} \to \omega_\nu$ making $(\mathcal{Y} \to B, \nu, \beta)$ a family of odd spin curves. The rest of the argument parallels the proof above. $\square$

Consider the universal curve $\pi : \mathcal{C} \to \overline{\mathcal{M}}_{g,2}$ with two sections $\sigma_1, \sigma_2 : \overline{\mathcal{M}}_{g,2} \to \mathcal{C}$. We will denote by $\overline{\mathcal{S}}_{g,2}(\omega_\pi(\sigma_1 + \sigma_2))$ the moduli space of square roots of the twisted canonical bundle $\omega_\pi(\sigma_1 + \sigma_2)$.

Definition 1.18. Let $\overline{T}_{g,2} = \overline{\mathcal{S}}_{g,2}(\omega_\pi(\sigma_1 + \sigma_2))$, where T stands for roots of the *twisted* canonical bundle.

By abuse of notation, let $\pi : \mathcal{C} \to \overline{T}_{g,2}$ continue to denote the universal curve and $\sigma_i : \overline{T}_{g,2} \to \mathcal{C}$ the two sections. Let $(\mathcal{E}, \alpha : \mathcal{E}^{\otimes 2} \to \omega_\pi)$ be the universal spin structure over $\mathcal{C}$. Let $\pi' : \mathcal{Y} \to \overline{T}_{g,2}$ be the curve obtained by gluing $\mathcal{C}$ along the two sections σ_1 and σ_2.

We wish to glue $\mathcal{E}$ along the two sections σ_i in order to descend $\mathcal{E}$ to a spin structure on $\mathcal{Y}$. The gluing is merely the choice of an isomorphism $\sigma_1^*\mathcal{E} \xrightarrow{\sim} \sigma_2^*\mathcal{E}$. However, it is not clear that these two pullbacks are isomorphic and has to be proven.

As σ_i is in the smooth locus of $\mathcal{C}$, the root $\mathcal{E}$ is locally free around σ_i and so the squaring map α is an isomorphism around σ_i. In particular, $\sigma_i^*\mathcal{E}$ is a square root of the line bundle $\sigma_i^*\omega_{\pi'}(\sigma_1 + \sigma_2)$. Since π' is smooth around σ_i and the base $\overline{T}_{g,2}$ is smooth we can apply adjunction formula to see that $\sigma_1^*\omega_{\pi'}(\sigma_1 + \sigma_2) \simeq \sigma_1^*(\omega_{\mathcal{C}} \otimes \pi'^{,*}\omega_{\overline{T}_{g,2}}(\sigma_1)) \simeq \mathcal{O}_{\overline{T}_{g,2}}$. The same computation works to show $\sigma_2^*\omega_{\pi'}(\sigma_1 + \sigma_2) \simeq \mathcal{O}_{\overline{T}_{g,2}}$. Hence $\sigma_1^*\mathcal{E}$ and $\sigma_2^*\mathcal{E}$ are both square roots of the trivial line bundle.

Proposition 1.19. *There exists a unique isomorphism $\psi : \sigma_1^*\mathcal{E} \xrightarrow{\sim} \sigma_2^*\mathcal{E}$ such that gluing $\mathcal{E}$ using ψ yields an odd spin structure on $\mathcal{Y} \to \overline{T}_{g,2}$. The gluing performed by $-\psi$ yields an even spin structure.*

Proof. On the universal curve $\pi : \mathcal{C} \to \overline{T}_{g,2}$ consider the following exact sequence:

$$0 \longrightarrow \mathcal{E}(-\sigma_i) \longrightarrow \mathcal{E} \longrightarrow \sigma_{i,*}\sigma_i^*\mathcal{E} \longrightarrow 0$$

Push this sequence forward to get:

$$0 \to \pi_*\mathcal{E}(-\sigma_i) \to \pi_*\mathcal{E} \to \sigma_i^*\mathcal{E} \to \mathrm{R}^1\pi_*\mathcal{E}(-\sigma_i) \to \mathrm{R}^1\pi_*\mathcal{E} \to 0$$

Because $\mathrm{R}^2\pi_* = 0$ on coherent sheaves, the first derived pushforwards satisfy base change. For any $(C, p, q) \in \overline{\mathcal{M}}_{g,2}$ we showed that $\forall \tau \in \sqrt{\omega_C(p+q)}$, $h^0(\tau - p) = h^0(\tau - p - q)$. Therefore $h^1(\tau - p) = h^0(\tau - q)$ and $h^1(\tau) = h^0(\tau - p - q)$ are equal. This implies that the surjective map $\mathrm{R}^1\pi_*\mathcal{E}(-\sigma_i) \to \mathrm{R}^1\pi_*\mathcal{E}$ must be an isomorphism. We are then left with the following exact sequence:

$$0 \longrightarrow \pi_*\mathcal{E}(-\sigma_i) \longrightarrow \pi_*\mathcal{E} \longrightarrow \sigma_i^*\mathcal{E} \longrightarrow 0$$

We claim that the image of $\pi_*\mathcal{E}(-\sigma_i)$ in $\pi_*\mathcal{E}$ coincides with the image of $\pi_*\mathcal{E}(-\sigma_1 - \sigma_2)$. Any local section t of $\pi_*\mathcal{E}(-\sigma_i)$ tensor squares to a local section $t^{\otimes 2}$ of $\pi_*\omega_\pi(\sigma_j - \sigma_i)$, $\{i, j\} = \{1, 2\}$. By residue theorem, applied in the relative context, $t^{\otimes 2}$ must also vanish along σ_j. As the base is reduced, and char $\neq 2$, this can happen only if t itself vanishes along σ_j. Therefore, t can be identified with a section of $\pi_*\mathcal{E}(-\sigma_1 - \sigma_2)$. This gives us the following exact sequence:

$$0 \longrightarrow \pi_*\mathcal{E}(-\sigma_1 - \sigma_2) \longrightarrow \pi_*\mathcal{E} \longrightarrow \sigma_i^*\mathcal{E} \longrightarrow 0$$

Since $\sigma_1^*\mathcal{E}$ and $\sigma_2^*\mathcal{E}$ are the cokernels of the same map, there is a canonical isomorphism $\psi : \sigma_1^*\mathcal{E} \xrightarrow{\sim} \sigma_2^*\mathcal{E}$ commuting with the surjections from $\pi_*\mathcal{E}$.

Let $\overline{\mathcal{E}}$ denote the torsion-free sheaf on $\pi' : \mathcal{Y} = \mathcal{C}/(\sigma_1 \sim \sigma_2) \to \overline{T}_{g,2}$ obtained by gluing $\mathcal{E}$ along the two sections via the isomorphism ψ. We claim that $\overline{\mathcal{E}}$ is an odd spin structure on $\mathcal{Y}$. If we had used $-\psi$ instead, we would have obtained an even spin structure as will be clear from the rest of the proof.

The relative dualizing sheaf $\omega_{\pi'}$ on $\mathcal{Y}$ pullsback on $\mathcal{C}$ to $\omega_\pi(\sigma_1+\sigma_2)$. As we demonstrated above we have $\sigma_i^*\omega_\pi(\sigma_1 + \sigma_2) \simeq \mathcal{O}_{\overline{T}_{g,2}}$ and up to scaling we can arrange it so that this isomorphism is the relative residue map, say $\mathrm{res}_i : \omega_\pi(\sigma_1+\sigma_2) \twoheadrightarrow \mathcal{O}_{\sigma_i}$. Then $\omega_{\pi'}$ is obtained by gluing $\omega_\pi(\sigma_1 + \sigma_2)$ along the two sections so that the following map commutes:

$$\begin{tikzcd}
 & \omega_\pi(\sigma_1+\sigma_2) \arrow[dl, "\mathrm{res}_1"'] \arrow[dr, "\mathrm{res}_2"] & \\
\mathcal{O}_B \arrow[rr, "-1"] & & \mathcal{O}_B
\end{tikzcd}$$

We have a map $\alpha : \mathcal{E}^{\otimes 2} \to \omega_\pi(\sigma_1+\sigma_2)$ which is an isomorphism near the sections. Squaring ψ will therefore will give a scalar isomorphism $\psi^{\otimes 2} : \mathcal{O}_B \to \mathcal{O}_B$, we claim that $\psi^{\otimes 2} = -1$. We can check the value of this scalar on any closed point $(C, p, q, \tau \in \sqrt{\omega_C(p+q)}) \in \overline{T}_{g,2}$. Here ψ is the isomorphism $\tau|_p \xrightarrow{\sim} \tau_q$ induced by picking a section $s \in \mathrm{H}^0(\tau) \setminus \mathrm{H}^0(\tau - p - q)$ and mapping $s|_p$ to $s|_q$. This is independent of any choices that are made. Moreover, since $s^{\otimes 2}$ is a meromorphic differential, i.e., $s^{\otimes 2} \in \mathrm{H}^0(\omega_C(p+q))$, it will satisfy the residue theorem. That is $\mathrm{res}_p(s^{\otimes 2}) = -\,\mathrm{res}_q(s^{\otimes 2})$. Since $\psi^{\otimes 2}$ must map $s^{\otimes 2}|_p$ to $s^{\otimes 2}|_q$ we must have $\psi^{\otimes 2} = -1$.

This implies that, using α, we can construct a map $\beta : \overline{\mathcal{E}}^{\otimes 2} \to \omega_{\pi'}$ which will make the triplet $(\mathcal{Y}, \overline{\mathcal{E}}, \beta)$ a family of spin curves. It remains to show that $\overline{\mathcal{E}}$ is an odd spin structure. As parity is invariant under families it will be sufficient to check the parity at a general closed point $(C, p, q, \tau) \in \overline{T}_{g,2}$.

Recall that $h^0(\tau) - h^0(\tau - p - q) = 1$ and by generality assumption we have $h^0(\tau - p - q) = 0$. We glued τ using a section $s \in \mathrm{H}^0(\tau)$ and identifying $s|_p$ with $s|_q$. Thus, $\overline{\mathcal{E}}$ restricts to $\tau/(s|_p \mapsto s|_q)$ on $C/(p \sim q)$. Under this gluing map the section s of τ survives and we continue to have 1 dimensional space of sections. This proves $\overline{\mathcal{E}}$ is an odd spin structure. If we had used $-\psi$ instead, then s would not have survived and the parity would have changed by 1, giving us an even spin structure. □

The proof above allows us to strengthen an observation of Cornalba [Cor89]. For any $(C, p, q) \in \mathcal{M}_{g,2}$ and any $\tau \in \sqrt{\omega_C(p+q)}$ we can pick a section $s \in \mathrm{H}^0(C, \tau)$ such that $s(p), s(q) \neq 0$. We then obtain an isomorphism $h : \tau|_p \xrightarrow{\sim} \tau|_q$ mapping $s(p)$ to $s(q)$. This isomorphism is independent of the choice of s and the line bundle $L = (\tau, h)$ on $X = C/(p \sim q)$ obtained by gluing τ via h gives a square root of ω_X. Thus, L is a theta characteristic. There is one other theta characteristic that can be obtained from τ; the line bundle $L' = (\tau, -h)$. Note that $h^0(X, L) = h^0(\tau)$ while $h^0(X, L') = h^0(\tau) - 1$. Therefore, one of L and L' is odd and the other one an even theta characteristic. All this has already been observed by Cornalba, but the following seems to be new: L is always odd and L' even.

Corollary 1.20. *For any $(C, p, q) \in \mathcal{M}_{g,2}$ and any $\tau \in \sqrt{\omega_C(p+q)}$ we have $h^0(\tau) \equiv 1 \pmod 2$. In particular, the theta characteristic L constructed above is always odd and L' is even.*

Proof. Immediate from the last part of the proof of Proposition 1.19. □

1.5 Boundary of the moduli of spin curves

Our goal here is to setup notation and briefly recall the properties of the stable spin curves. See Cornalba [Cor89] for the proof of the statements below. We will, however, deviate from his notation in favor of what we believe to be a more efficient one. See Section 1.5.1 for a dictionary between the two notations.

Definition 1.21. Let $\Delta_i^+ \subset \overline{\mathcal{S}}_g^-$ be the closed substack defined as the image of the clutching map $\overline{\mathcal{S}}_{i,1}^+ \times \overline{\mathcal{S}}_{g-i,1}^- \to \overline{\mathcal{S}}_g^-$. We also define Δ_i^- as being equal to Δ_{g-i}^+.

Take a stable curve of genus g, say $X \in \Delta_i \subset \overline{\mathcal{M}}_g$ with $i \in \{1, \ldots, g-1\}$. Assuming X is sufficiently general in Δ_i we can write $X = C_1 \cup_{p \sim q} C_2$ with $g(C_1) = i$. A theta characteristic on X can be uniquely determined, up to isomorphism, by giving theta characteristics on C_1 and C_2 (see Example 1.12).

If we wish to end up with an odd theta characteristic on X then the theta characteristics on C_1 and C_2 must have opposite parities, therefore it will be sufficient to specify the parity on C_1 as the parity on C_2 can be inferred. Such curves, and their degenerations, form the closed points of the boundary components Δ_i^+ and Δ_i^- with the sign denoting the parity of the spin structure on the genus i component.

Definition 1.22. Let $\Delta_0^b \subset \overline{\mathcal{S}}_g^-$ be the closed substack defined as the image of the clutching map $\overline{\mathcal{S}}_{g-1,2}^- \to \overline{\mathcal{S}}_g^-$.

We defined $\Delta_0^b \subset \overline{\mathcal{S}}_g^-$ so that the closed points of Δ_0^b correspond to spin curves with a node of type 0. Here b stands for *blow-up*, as passage to the quasi-stable model blows-up the node by inserting a $\mathbb{P}^1$ there.

Definition 1.23. Let $\Delta_0^n \subset \overline{\mathcal{S}}_g^-$ be the closed substack defined as the image of the clutching map $\overline{\mathcal{S}}_{g,2}(\omega_\pi(\sigma_1+\sigma_2)) \to \overline{\mathcal{S}}_g^-$.

We defined $\Delta_0^n \subset \overline{\mathcal{S}}_g^-$ to be the locus of spin curves whose underlying curve admits type 0 nodes which are not singularities of the spin structure. Here n stands for *no blow-up*, because the quasi-stable model will not blow-up the node.

Proposition 1.24. *The boundary of $\overline{\mathcal{S}}_g^-$ decomposes into the following union where each term on the right is irreducible:*

$$\overline{\mathcal{S}}_g^- \setminus \mathcal{S}_g^- = \Delta_0^b \cup \Delta_0^n \cup \bigcup_{i=1}^{g-1} \Delta_i^+.$$

Proof. This result is originally due to Cornalba [Cor89]. But we provide a sketch. The fact that each boundary component Δ_i^x is irreducible follows from the fact that they are the images of irreducible moduli spaces. To see that this is the entirety of the boundary one can repeat the argument in Section 1.9 where we compute the degree over each $\Delta_i \subset \overline{\mathcal{M}}_g$. This is of course much simpler in this context with a single theta characteristic. □

It is convenient to break the union above into pieces as follows. Let $\pi : \overline{\mathcal{S}}_g^- \to \overline{\mathcal{M}}_g$ be the forgetful map and notice that for each $i = 1, \ldots, g-1$ we have

$$\begin{aligned}\pi^{-1}(\Delta_i) &= \Delta_i^+ \cup \Delta_{g-i}^+ \\ &= \Delta_i^+ \cup \Delta_i^-.\end{aligned}$$

Note however that if $i = \frac{g}{2}$ then $\Delta_{\frac{g}{2}}^+ = \Delta_{\frac{g}{2}}^-$ and so $\pi^{-1}(\Delta_{\frac{g}{2}}) = \Delta_{\frac{g}{2}}^+$ is irreducible. When $i = 0$ we have

$$\pi^{-1}(\Delta_0) = \Delta_0^n \cup \Delta_0^b.$$

Remark 1.25. In fact, $\pi^{-1}(\Delta_i)$ are non-reduced when working with stacks although $\Delta_i^\pm$ are reduced. We ignored this fact in the formulae above for readability. The multiplicities are taken into account when dealing with divisor classes, see Equation 1.8.2.

1.5.1 Dictionary

In the paper [Cor89] of Cornalba, the boundary components of $\overline{\mathcal{S}}_g^-$ are denoted by A_i^- and B_i^-, whereas the boundary components of $\overline{\mathcal{S}}_g^+$ are denoted by A_i^+ and B_i^+. We provide a table below for quick translation between his notation and ours.

Note that, when g is even Cornalba defines $B_{\frac{g}{2}}^\pm$ to be the empty set. On the other hand, we have $\Delta_i^+ = \Delta_{g-i}^-$ so the most convenient notation can be chosen. See for example Equation 1.5.1 below.

$\overline{\mathcal{S}}_g^-$		
Δ_0^b		B_0^-
Δ_0^n		A_0^-
Δ_{g-1}^-	Δ_1^+	A_1^-
Δ_{g-2}^-	Δ_2^+	A_2^-
$\vdots$	$\vdots$	$\vdots$
Δ_2^-	Δ_{g-2}^+	B_2^-
Δ_1^-	Δ_{g-1}^+	B_1^-

$\overline{\mathcal{S}}_g^+$		
Δ_0^b		B_0^+
Δ_0^n		A_0^+
Δ_{g-1}^-	Δ_1^+	A_1^+
Δ_{g-2}^-	Δ_2^+	A_2^+
$\vdots$	$\vdots$	$\vdots$
Δ_2^-	Δ_{g-2}^+	B_2^+
Δ_1^-	Δ_{g-1}^+	B_1^+

We were compelled to refine the convention set by Cornalba because its minor idiosyncrasies are compounded when used with double spin curves, which has many more boundary components, leading to unmanageable formulae. We sought to make the following improvements:

1. Better use of superscripts. The difference between A_i^+ and A_i^- need not be highlighted. The geometry of $\overline{\mathcal{S}}_g^-$ and $\overline{\mathcal{S}}_g^+$ are very different and any class computation will leave no doubt as to which space is being considered. So we put the superscript to better use.

2. Intuitive notation. The choice of the letters A and B is arbitrary as well as the ordering they imply. We feel Δ_i^+ and Δ_i^- are unambiguous.

3. Symmetry breaking. If g is even $B_{\frac{g}{2}}$ is defined to be the empty set whereas $\Delta_{\frac{g}{2}}^-$ means exactly what it should.

4. Shorter formulae. Better use of symmetry usually halves the number of summations required to express a divisor. See the example below.

The classes of the above mentioned divisors are denoted by their corresponding small Greek letters: $\delta_i^+ := [\Delta_i^+]$ and $\alpha_i^+ := [A_i^+]$ and so on. As an example, we take the divisor class formula for $Z_g \subset \overline{\mathcal{S}}_g^-$ from [FV14]. Recall that Z_g is the closure of the locus $\{(C, \eta) \mid \exists p \in C,\, 2p \leq \eta\}$.

The superscripts from α's and β's are omitted in [FV14] and we will do so as well. In addition, the α_i and β_i of [FV14] is $2\alpha_i$ and $2\beta_i$ of Cornalba, except α_0's agree. After translating the formula for $[Z_g]$ into the notation of Cornalba and then into our notation we get:

$$\begin{aligned}
[Z_g] &= (g+8)\lambda - \frac{g+2}{4}\alpha_0 - 4\beta_0 - \sum_{i=1}^{\lfloor \frac{g}{2} \rfloor} 4(g-i)\alpha_i - \sum_{i=1}^{\lfloor \frac{g}{2} \rfloor} 4i\beta_i \\
&= (g+8)\lambda - \frac{g+2}{4}\delta_0^n - 4\delta_0^b - \sum_{i=1}^{g-1} 4i\delta_i^- \qquad (1.5.1)
\end{aligned}$$

1.6 Degenerate double spin curves

Before we define the boundary components of $\overline{\mathcal{S}}_g^{--}$, let us give examples that motivate the names of these components. For the definition of a *family* of double spin curves we will refer to Part I, however our examples should mostly be clear by themselves.

1.6.1 Stable models of double spin curves

For the following four examples let $\overline{X} = C/(p \sim q)$ and let $\nu : C \to \overline{X}$ be the normalization map.

Example 1.26 (Generic curve in Δ_0^{bb})**.** Let η_1, η_2 be distinct theta characteristics on C. Define $\mathcal{E}_i = \nu_* \eta_i$. This time the synchronization of the pair $(\mathcal{E}_1, \mathcal{E}_2)$ is a non-trivial matter. From the point of view of synchronizations, Remark I.2.24 or Lemma 1.38 implies that there are precisely two non-isomorphic synchronizations. In favor of notational simplicity, we will denote these synchronized tuples by $(\eta_1, \eta_2)'$ and $(\eta_1, \eta_2)''$. The double spin curves $(\overline{X}, (\eta_1, \eta_2)')$ and $(\overline{X}, (\eta_1, \eta_2)'')$ belong to Δ_0^{bb}.

Example 1.27 (Generic curve in $\Delta_0^{b=}$)**.** This time let $\mathcal{E}_1 = \mathcal{E}_2 = \nu_* \eta$ for some theta characteristic on C. One of the synchronizations will simply be the square of an isomorphism $\mathcal{E}_1 \xrightarrow{\sim} \mathcal{E}_2$ and the other will not be. The first of these is naturally the limit of a pair of isomorphic theta characteristics, something which does not belong to the moduli space $\overline{\mathcal{S}}_g^{--}$. Therefore, there is only one synchronization between $\mathcal{E}_1$ and $\mathcal{E}_2$ that makes the resulting double spin curve an element of $\overline{\mathcal{S}}_g^{--}$. We will denote this double spin curve by $(\overline{X}, (\eta, \eta)')$. Such objects will form the generic elements of the boundary component $\Delta_0^{b=}$.

Example 1.28 (Generic curves in Δ_0^{bn} and Δ_0^{nb})**.** Let η be a theta characteristic on C. Moreover, let L be a line bundle on L squaring to $\omega_{\overline{X}}$ and with $h^0(L) \equiv 1 \pmod 2$. Then the tuple $(\nu_* \eta, L)$ requires no synchronization as the second limit theta characteristic is free at the node. Spin curves of the form $(\overline{X}, \nu_* \eta, L)$ belongs to the boundary component $\Delta_0^{bn} \subset \overline{\mathcal{S}}_g^{--}$ and $(\overline{X}, L, \nu_* \eta)$ belongs to the boundary component Δ_0^{nb}.

Example 1.29 (Generic curve in Δ_0^{nn})**.** Finally, consider two distinct line bundles L_1 and L_2 on $\overline{X}$ such that $L_i^{\otimes 2} \simeq \omega_{\overline{X}}$. Then the tuple $(\overline{X}, L_1, L_2)$ is a double spin curve and if $h^0(L_i) \equiv 1 \pmod 2$ then the spin structures are odd. Such double spin curves belong to the boundary component Δ_0^{nn}.

Let $\overline{X} = C_1 \cup_{p \sim q} C_2$ and let η_{ij} be a theta characteristic on C_i with $i, j \in \{1, 2\}$. Let $\Xi = \left(\begin{smallmatrix} \eta_{11} & \eta_{12} \\ \eta_{21} & \eta_{22} \end{smallmatrix}\right)$ and assume that the tuples (η_{11}, η_{12}) and (η_{21}, η_{22}) are distinct.

Example 1.30 (Generic curve in Δ_i^{xy}, $i > 0$)**.** The pair $(\overline{X}, \Xi)$ determines the isomorphism class of a double spin structure on $\overline{X}$ in the following way. For $i = 1, 2$ let $\sigma_i^* \mathcal{E} = \iota_{1,*} \eta_{i1} \oplus \iota_{2,*} \eta_{i2}$ be the two theta characteristics obtained on $\overline{X}$ as in Section 1.3. We simply need to synchronize $\mathcal{E}_1$ and $\mathcal{E}_2$. However, as each η_{ij} can be scaled by ± 1 independently to yield an isomorphism of $\sigma_i^* \mathcal{E}$, any two synchronizations are identified up to isomorphism.

Remark 1.31. Since we want $(\overline{X}, \Xi)$ to be a double *odd* spin curve the η_{i1} and η_{i2} must have opposite parities. The parities alone give 4 distinct cases and when $\eta_{l1} \not\simeq \eta_{l2}$ for $l = 1, 2$ such curves will form the generic members of the boundary components $\Delta_i^{++}, \Delta_i^{+-}, \Delta_i^{-+}, \Delta_i^{--}$ where the superscript denotes the parity of the two theta characteristics on C_1 and $i = g(C_1)$. We also need to consider the cases where $\eta_{11} \simeq \eta_{21}$ or $\eta_{12} \simeq \eta_{22}$. Such curves will form generic elements in $\Delta_i^{+=}, \Delta_i^{-=}$ or $\Delta_{g-i}^{+=}, \Delta_i^{-=}$ respectively.

1.6.2 Quasi-stable models of double spin curves

The passage from the stable model to the quasi-stable model is slightly different for double spin curves than for spin curves, so we recall the procedure here. Let $(\overline{X}, \mathcal{E}_1, \mathcal{E}_2)$ be a double spin curve. Define $X_i = \underline{\mathrm{Proj}}(\mathrm{Sym}^* \mathcal{E}_i)$ and $\mathcal{O}_{X_i}(1)$ as in Section 1.3. In all the examples above, except for those of type Δ_0^{nb} and Δ_0^{bn}, the synchronization data is equivalent to an isomorphism $\mathrm{Sym}^{2*} \mathcal{E}_1 \xrightarrow{\sim} \mathrm{Sym}^{2*} \mathcal{E}_2$. Let $X = \underline{\mathrm{Proj}}(\mathrm{Sym}^{2*} \mathcal{E}_i)$ and $f_i : X \xrightarrow{\sim} X_i$ be the natural isomorphisms, each of which satisfy $f_i^* \mathcal{O}_{X_i}(1)^{\otimes 2} \simeq \mathcal{O}_X(1)$. With $L_i := f_i^* \mathcal{O}_{X_i}(1)$, the quasi-stable model of $(\overline{X}, \mathcal{E}_1, \mathcal{E}_2)$ is (X, L_1, L_2). Recall also that there is the structure map of the $\underline{\mathrm{Proj}}$ construction, $\pi : X \to \overline{X}$, collapsing any exceptional components.

We now list the end result of this construction for each of the examples above, continuing to use the notation introduced in each of the corresponding examples. For the next three examples we have $X = C \cup_{\substack{p\sim 0\\ q\sim\infty}} \mathbb{P}^1$. We emphasize that the $\underline{\mathrm{Proj}}$ construction equips X with a line bundle $\mathcal{O}_X(1)$ which is to be viewed as the *square* of L_i's.

Example 1.32 (Generic curve in Δ_0^{bb})**.** We have $L_i|_C \simeq \eta_i$ and $L_i|_{\mathbb{P}^1} \simeq \mathcal{O}_{\mathbb{P}^1}(1)$. Let us elaborate on the subtleties involved. There are two different synchronizations of the theta characteristics, both yield isomorphic quasi-stable models which we identify with X, but non-isomorphic double spin structures on X. Fixing $\mathcal{O}_X(1)$, there will be two line bundles squaring to $\mathcal{O}_X(1)$ and pulling back to η_i on C, so we denote these by L'_{η_i} and L''_{η_i}. The double spin curves $(X, L'_{\eta_1}, L'_{\eta_2})$ and $(X, L''_{\eta_1}, L''_{\eta_2})$ are isomorphic but distinct from the isomorphic pair $(X, L'_{\eta_1}, L''_{\eta_2})$ and $(X, L''_{\eta_1}, L'_{\eta_2})$.

Example 1.33 (Generic curve in Δ_0^{be})**.** This time $L_i|_C \simeq \eta$ and $L_i|_{\mathbb{P}^1} \simeq \mathcal{O}_{\mathbb{P}^1}(1)$, independent of i. It is worth mentioning that $L_1 \not\simeq L_2$. As in previous example let us denote by L'_η and L''_η the two roots of $\mathcal{O}_X(1)$ restricting to η on C. Then the quasi-stable model is (X, L'_η, L''_η) which is isomorphic to (X, L''_η, L'_η). The double spin curves (X, L'_η, L'_η) and (X, L''_η, L''_η) do not appear in $\overline{\mathcal{S}}_g^{--}$.

Example 1.34 (Generic curves in Δ_0^{bn} and Δ_0^{nb})**.** We have $L_1|_C \simeq \eta$ and $L_1|_{\mathbb{P}^1} \simeq \mathcal{O}_{\mathbb{P}^1}(1)$ whereas $L_2 \simeq \pi^* L$.

Example 1.35 (Generic curve in Δ_0^{nn})**.** In this case, $X = \overline{X}$ and the stable model and the quasi-stable model coincide.

For the final example let $\overline{X} = C_1 \cup_{p\sim q} C_2$ with $g(C_1) = i$, $1 \le i \le g - 1$. Then $X = C_1 \cup_{p\sim 0} \mathbb{P}^1 \cup_{\infty\sim q} C_2$.

Example 1.36 (Generic curve in Δ_i^{xy}, i >0)**.** We have $L_i|_{C_j} \simeq \eta_{ij}$ and $L_i|_{\mathbb{P}^1} \simeq \mathcal{O}_{\mathbb{P}^1}(1)$ for $i, j = 1, 2$. Since X is of compact type, the gluing data required for the definition of L_i do not change the isomorphism classes of L_i's. Hence, we may as well denote the quasi stable model by (X, Ξ) with $\Xi = (\eta_{ij})_{i,j}$ as before.

Remark 1.37. The notational convention regarding x and y is as explained in Remark 1.31.

1.7 Clutching morphisms, II

Let $\mathcal{X} \to B$ be a family of nodal curves obtained by gluing another, possibly disconnected, family of curves $\mathcal{Y} \to B$ along two disjoint sections $\sigma_1, \sigma_2 : B \to \mathcal{Y}$. Denote the gluing map by $\nu : \mathcal{Y} \to \mathcal{X}$. For $i = 1, 2$ let $(\mathcal{L}_i, \alpha_i : \mathcal{L}_i^2 \to \omega_{\mathcal{Y}/B})$ be distinct spin structures on $\mathcal{Y}$ (or, on each component of $\mathcal{Y}$) and suppose that these spin structures are synchronized. In a Zariski open subset $V \subset \mathcal{Y}$ containing σ_1 and σ_2 the synchronization yields an isomorphism $\psi : \mathcal{L}_1^2|_V \xrightarrow{\sim} \mathcal{L}_2^2|_V$. Let $\mathcal{E}_i = \nu_* \mathcal{L}_i$ be the resulting torsion-free roots on $\omega_{\mathcal{X}/B}$ as in Section 1.4.

Lemma 1.38. *The spin structures $\mathcal{E}_1$ and $\mathcal{E}_2$ can be synchronized iff $\sigma_1^* \mathcal{L}_1 \otimes \sigma_2^* \mathcal{L}_1 \simeq \sigma_1^* \mathcal{L}_2 \otimes \sigma_2^* \mathcal{L}_2$.*

Proof. Let $\sigma : B \to \mathcal{X}$ be the image of σ_i's. Necessarily $\mathcal{Y}$ is smooth along σ_1 and σ_2. As a synchronization concerns only a neighbourhood of σ, we may assume $\mathcal{Y}$ is smooth over B for the sake of notational simplicity and without loss of generality.

For any torsion-free root $\mathcal{E} = \nu_* \mathcal{L}$ on $\mathcal{X}$ we have $\operatorname{Sym}^2 \mathcal{E} \simeq \nu_*(\mathcal{L}^2) \oplus \sigma_*(\sigma_1^* \mathcal{L} \otimes \sigma_2^* \mathcal{L})$. Indeed, $\mathcal{L}$ is necessarily locally free near σ_i's and the generators of $\mathcal{L}$ near σ_1 and σ_2 tensor together to give a pure torsion element, supported only at σ. This is the torsion part of $\operatorname{Sym}^2 \mathcal{E}$. In particular, a pre-sync data $\Psi : \operatorname{Sym}^2 \mathcal{E}_1 \xrightarrow{\sim} \operatorname{Sym}^2 \mathcal{E}_2$ compatible with ψ determines, and is determined by, an isomorphism $u : \sigma_1^* \mathcal{L}_1 \otimes \sigma_2^* \mathcal{L}_1 \xrightarrow{\sim} \sigma_1^* \mathcal{L}_2 \otimes \sigma_2^* \mathcal{L}_2$. Indeed, given any such u simply define $\Psi = \nu_* \psi \oplus u$ which is clearly an isomorphism between $\operatorname{Sym}^2 \mathcal{E}_1 \xrightarrow{\sim} \operatorname{Sym}^2 \mathcal{E}_2$.

It remains to check the conditions on u for which Ψ is a sync data. This means that the map:

$$\operatorname{Sym}^2 \operatorname{Sym}^2 \mathcal{E}_1 \to \operatorname{Sym}^4 \mathcal{E}_2,$$

induced by Ψ, factors through $\operatorname{Sym}^4 \mathcal{E}_1$. Let ξ_{ij} be a local generator of $\mathcal{L}_i$ near σ_j. Then the kernel $\operatorname{Sym}^2 \operatorname{Sym}^2 \mathcal{E}_1 \to \operatorname{Sym}^4 \mathcal{E}_1$ is locally generated by $\xi_{11}^2 \xi_{12}^2 - (\xi_{11}\xi_{12})^2$. So we need only check when this kernel is annihilated by Ψ. The kernel is annihilated iff it is annihilated after passing to an étale neighbourhood of σ. Therefore, we may suppose ξ_{ij}'s are chosen so that $\psi(\xi_{1j}^2) = \xi_{2j}^2$.

In any case, we have $u(\xi_{11}\xi_{12}) = c \cdot \xi_{21}\xi_{22}$ for some $c \in k^*$. Now we observe $\Psi(\xi_{11}^2 \xi_{12}^2 - (\xi_{11}\xi_{12})^2) = \xi_{21}^2 \xi_{22}^2 - c^2 (\xi_{21}\xi_{22})^2$. Therefore, Ψ is a sync data precisely when $c = \pm 1$. Clearly, $c = \pm 1$ iff $u^2 = \sigma_1^* \psi \otimes \sigma_2^* \psi$. ◻

Corollary 1.39. *Suppose $\mathcal{E}_1$ and $\mathcal{E}_2$ can be synchronized. If $\mathcal{Y}$ is connected, then up to isomorphism there are two distinct synchronization between $\mathcal{E}_1$ and $\mathcal{E}_2$. If $\mathcal{Y}$ is disconnected, so that $\mathcal{Y}$ has two components, then up to isomorphism there is only one synchronization between $\mathcal{E}_1$ and $\mathcal{E}_2$.*

Proof. Since $\mathcal{X} \to B$ is a family of stable curves, if $\mathcal{Y}$ has two components then $\mathcal{Y} = \mathcal{Y}_1 \sqcup \mathcal{Y}_2$ where $\mathcal{Y}_i$ contains the section σ_i. Let $\mathcal{L}_{ij} = \mathcal{L}_i|_{\mathcal{Y}_j}$ and observe that scaling $\mathcal{L}_{ij}$ by ± 1 will give automorphisms compatible with the sync data ψ. However, as the proof above demonstrates, scaling $\mathcal{L}_{11}$ by -1 while scaling the other $\mathcal{L}_{ij}$'s by 1 will descend to an isomorphism between the two possible synchronizations of $\mathcal{E}_1$ and $\mathcal{E}_2$.

If $\mathcal{Y}$ is connected, then such scalings will act on both the domain and range of u, see the proof of Lemma 1.38. There are no other isomorphisms of $\mathcal{E}_i$'s and so the two synchronizations must be non-isomorphic. □

Remark 1.40. These two results give another proof of the argument made in Example 1.26 that there are two non-isomorphic spin structures when $X \to \operatorname{Spec} k$ is an irreducible nodal curve for any given pair of spin structures on its normalization $\nu : C \to X$.

1.7.1 Constructing the clutching maps

Clutching for Δ_i^{xy}.

Now we will define a clutching map $\overline{\mathcal{S}}_{i,1}^{++} \times \overline{\mathcal{S}}_{g-i,1}^{--} \to \overline{\mathcal{S}}_g^{--}$. Let $B = \overline{\mathcal{S}}_{i,1}^{++} \times \overline{\mathcal{S}}_{g-i,1}^{--}$ which comes equipped with two universal curves $\pi_1 : \mathcal{C}_1 \to B$ and $\pi_2 : \mathcal{C}_2 \to B$ of genera i and $g-i$ respectively. Let $\sigma_i : B \to \mathcal{C}_i$ denote the universal section on each curve and let $\mathcal{X} \to B$ be obtained by gluing $\mathcal{C}_1$ and $\mathcal{C}_2$ together along the sections. Let $\nu : \mathcal{C}_1 \sqcup \mathcal{C}_2 \to \mathcal{X}$ be the normalization map.

Each $\mathcal{C}_i$ will come with a pair of (synchronized) spin structures $(\mathcal{L}_{i1}, \mathcal{L}_{i2})$. Let $A_i = \sigma_1^*\mathcal{L}_{i1} \otimes \sigma_2^*\mathcal{L}_{i2}$ and $\mathcal{E}_i = \nu_*(\mathcal{L}_{i1} \oplus \mathcal{L}_{i2})$. Recall that $\mathcal{E}_1$ and $\mathcal{E}_2$ can be synchronized iff $A_1 \simeq A_2$.

Since $A_i^{\otimes 2} \simeq \sigma_1^*\omega_{\mathcal{C}_1/B} \otimes \sigma_2^*\omega_{\mathcal{C}_2/B}$ for each i, the line bundle $\varepsilon := A_1 \otimes A_2^\vee$ squares to the identity. We may consider $\mathcal{L}'_{11} := \mathcal{L}_{11} \otimes \pi_1^*\varepsilon$. The family of theta characteristics defined by $\mathcal{L}'_{11}$ is equivalent to the original one defined by $\mathcal{L}_{11}$. Therefore, replace $\mathcal{L}_{11}$ by $\mathcal{L}'_{11}$ and $\mathcal{E}_1$ by $\mathcal{E}'_1 = \nu_*(\mathcal{L}'_{11} \oplus \mathcal{L}_{12})$. Now the two spin structures $\mathcal{E}'_1$ and $\mathcal{E}_2$ can be synchronized. We simply fix one synchronization, which we will omit from notation, as Corollary 1.39 implies that it is unique up to isomorphism.

Having defined a double odd spin structure on the genus g curve $\mathcal{X} \to B$, we constructed the clutching map $B \to \overline{\mathcal{S}}_g^{--}$ that we wanted.

The same argument may be applied to construct a clutching map with slightly different domains. Almost without modification, we can define a clutching map from $\overline{\mathcal{S}}_{i,1}^{-+} \times \overline{\mathcal{S}}_{g-i,1}^{+-} \to \overline{\mathcal{S}}_g^{--}$ for instance. We can also define a clutching map $\overline{\mathcal{S}}_{i,1}^{+} \times \overline{\mathcal{S}}_{g-i,1}^{--} \to \overline{\mathcal{S}}_g^{--}$ by using two copies of the even theta characteristic on $\mathcal{C}_1$.

Clutching for Δ_0^{bb}.

We would now wish to construct the clutching map from $\overline{\mathcal{S}}_{g-1,2}^{--}$, however Corollary 1.39 implies that this should be impossible: we need, in addition to the pair of theta characteristics, the choice of a synchronization. Indeed, let $B = \overline{\mathcal{S}}_{g-1,2}^{--}$ and $\mathcal{C} \to B$ be the universal curve with $\sigma_1, \sigma_2 : B \to \mathcal{C}$ the two sections. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be the two (synchronized) theta characteristic on $\mathcal{C}$. As before let $A_i = \sigma_1^*\mathcal{L}_i \otimes \sigma_2^*\mathcal{L}_i$ for $i = 1, 2$ and let $\varepsilon = A_1 \otimes A_2^\vee$ which is a 2-torsion line bundle.

Remark 1.41. By constructing test curves, for instance, it is possible to see that ε is not the trivial 2-torsion bundle. Glue $\mathcal{C}$ along σ_1 and σ_2 to obtain $\nu : \mathcal{C} \to \mathcal{X}$. Since $\varepsilon \neq \mathcal{O}_B$, the two spin structures $\nu_*\mathcal{L}_1$ and $\nu_*\mathcal{L}_2$ on $\mathcal{X}$ can not be synchronized by Lemma 1.38.

Let $B' \to B$ be the double cover of B corresponding to ε and pullback the double spin curve onto B', we will denote this by $(\mathcal{C}' \to B', \mathcal{L}_1', \mathcal{L}_2')$. Let $\nu' : \mathcal{C}' \to \mathcal{X}'$ be obtained by gluing the two sections $\sigma_1', \sigma_2' : B' \to \mathcal{C}'$. This time the two spin structures $\mathcal{E}_i = \nu_*'\mathcal{L}_i$, $i = 1, 2$, can be synchronized and we pick one synchronization. Changing the synchronization would correspond to the involution of B' over B, so any of the two choices is good enough.

This construction induces the clutching map $(\overline{\mathcal{S}}_{g-1,2}^{--})' \to \overline{\mathcal{S}}_g^{--}$, where the prime denotes the double cover we just constructed.

Clutching for $\Delta_0^{b=}$.

There is a clutching map $\overline{\mathcal{S}}_{g-1,2}^{-} \to \overline{\mathcal{S}}_g^{--}$, which is rather subtle. Let $B = \overline{\mathcal{S}}_{g-1,2}^{-}$ and let $(\mathcal{C} \to B, \mathcal{L})$ be the universal spin curve with markings $\sigma_1, \sigma_2 : B \to \mathcal{C}$. We let $\mathcal{L}_1 = \mathcal{L}_2 = \mathcal{L}$ and consider $\mathcal{C}$ with a pair of identical odd spin structures $(\mathcal{L}_1, \mathcal{L}_2)$. This time $A_1 \simeq A_2$ but what is more, there is a distinguished isomorphism between A_1 and A_2: the identity. If we pick the identity to be our synchronization then we do not get a map to $\overline{\mathcal{S}}_g^{--}$ however if we pick negative identity then we do.

Clutching for Δ_0^{bn} and Δ_0^{nb}.

If one of the two spin structures is free along a node, then there is a unique synchronization. This makes the remaining constructions particularly easy. Using our existing clutching morphisms for single theta characteristics, we construct the clutching map $\mathcal{S}_{g-1,2}^{-} \times_{\mathcal{M}_g} \mathcal{S}_{g-1,2}(\omega_\pi(\sigma_1 + \sigma_2)) \to \overline{\mathcal{S}}_g^{--}$. As we will take the closed image anyway, we need not have the compactified moduli spaces in the domain. The image of this clutching map corresponds to Example 1.28, that is to the boundary component Δ_0^{bn}. Changing the order of the product gives Δ_0^{nb}.

Clutching for Δ_0^{nn}.

Finally we can define the clutching map $\mathcal{S}_{g-1,2}(\omega_\pi(\sigma_1+\sigma_2)) \times_{\mathcal{M}_g} \mathcal{S}_{g-1,2}(\omega_\pi(\sigma_1+\sigma_2)) \setminus \Delta \to \overline{\mathcal{S}}_g^{--}$, where Δ is the diagonal of this self product. The image of this clutching map will contain the type of curves discussed in Example 1.29.

1.7.2 Defining the boundary components

Assume $g \geq 3$ throughout, as $g = 1, 2$ require a slightly different treatment.

Definition 1.42. For $x \in \{+, -\}$ let $\bar{x} \in \{+, -\}$ be such that $x \neq \bar{x}$.

Definition 1.43. For $i = 1, \dots, g-1$ and $x, y \in \{+, -\}$ let Δ_i^{xy} be the image of the clutching map $\overline{\mathcal{S}}_{i,1}^{xy} \times \overline{\mathcal{S}}_{g-i,1}^{\bar{x}\bar{y}} \to \overline{\mathcal{S}}_g^{--}$. Similarly define $\Delta_i^{x=}$ to be the image of the clutching map $\overline{\mathcal{S}}_{i,1}^{x} \times \overline{\mathcal{S}}_{g-i,1}^{\bar{x}\bar{x}} \to \overline{\mathcal{S}}_g^{--}$.

Definition 1.44. Let Δ_0^{bb} be the image of the clutching map $(\overline{\mathcal{S}}_{g-1,2}^{--})' \to \overline{\mathcal{S}}_g^{--}$ defined in the previous section. Let $\Delta_0^{b=}$ be the image of the clutching map $\overline{\mathcal{S}}_{g-1,2}^{-} \to \overline{\mathcal{S}}_g^{--}$, with two identical odd theta characteristics but with a non-trivial synchronization. Let Δ_0^{bn} be the image of $\mathcal{S}_{g-1,2}^{-} \times_{\mathcal{M}_g} T_{g-1,2} \to \overline{\mathcal{S}}_g^{--}$, Δ_0^{nb} be the image of $T_{g-1,2} \times_{\mathcal{M}_g} \mathcal{S}_{g-1,2}^{-} \to \overline{\mathcal{S}}_g^{--}$ and Δ_0^{nn} the image of $T_{g-1,2} \times_{\mathcal{M}_g} T_{g-1,2} \setminus \Delta \to \overline{\mathcal{S}}_g^{--}$.

Proposition 1.45. *With $\pi : \overline{\mathcal{S}}_g^{--} \to \overline{\mathcal{M}}_g$ denoting the forgetful map, the pullbacks of the boundary components decompose as follows:*

$$\pi^{-1}(\Delta_0) = \Delta_0^{bb} \cup \Delta_0^{b=} \cup \Delta_0^{bn} \cup \Delta_0^{nb} \cup \Delta_0^{nn}$$

and for $i = 1, \dots, g-1$:

$$\pi^{-1}(\Delta_i) = \Delta_i^{++} \cup \Delta_i^{+-} \cup \Delta_i^{+=} \cup \Delta_i^{-=} \cup \Delta_{g-i}^{++} \cup \Delta_{g-i}^{+-} \cup \Delta_{g-i}^{+=} \cup \Delta_{g-i}^{-=}.$$

Proof. We built these boundary components by considering degenerate theta characteristics case by case, exhausting the possibilities. This is one way to prove this proposition. However, as the boundary component $\Delta_0^{b=}$ exemplifies, some of these cases are tricky to spot. Therefore, we will give another proof in Section 1.9. $\square$

Remark 1.46. In fact, the pullbacks $\pi^{-1}(\Delta_i)$ must have been non-reduced, however we ignored this fact for readability. For the non-reduced structure see the pullback formulas in Section 1.8.3.

Remark 1.47. When $i = 1$ the fact that a genus 1 curve has a unique odd theta characteristic means that $\overline{\mathcal{S}}_{1,1}^{--} = \emptyset$. Therefore, $\Delta_{g-1}^{++} = \emptyset$ and $\Delta_{g-1}^{+=} = \emptyset$. This gives 6 non-empty boundary components over Δ_1:

$$\pi^{-1}(\Delta_1) = \Delta_1^{++} \cup \Delta_1^{+-} \cup \Delta_{g-1}^{+-} \cup \Delta_1^{+=} \cup \Delta_1^{-=} \cup \Delta_{g-1}^{-=}.$$

Remark 1.48. When g is even and $i = \frac{g}{2}$ we have $\Delta_{\frac{g}{2}}^{++} = \Delta_{\frac{g}{2}}^{--}$ so the union given in Proposition 1.45 collapses to:

$$\pi^{-1}(\Delta_{\frac{g}{2}}) = \Delta_{\frac{g}{2}}^{++} \cup \Delta_{\frac{g}{2}}^{+-} \cup \Delta_{\frac{g}{2}}^{+=} \cup \Delta_{\frac{g}{2}}^{-=}.$$

Remark 1.49. When $i \neq \frac{g}{2}$ and $i \neq 1$ we get 8 distinct boundary components lying over $\Delta_i \subset \overline{\mathcal{M}}_g$. Therefore, assuming $g \geq 3$, there are a total of $4g - 1$ distinct boundary components in $\overline{\mathcal{S}}_g^{--}$.

1.7.3 All these boundary components are irreducible

We will prove in Chapter 4 that each of the moduli spaces $\mathcal{S}_g^{--}$, $\mathcal{S}_g^{-+}$, $\mathcal{S}_g^{+-}$, $\mathcal{S}_g^{++}$ are irreducible for any g. Thus so will their closure and their product over $\operatorname{Spec} k$. This implies that the clutching morphisms have irreducible images Δ_i^{xy}, since their domains are irreducible.

In defining Δ_0^{bb} we used an étale, non-trivial double cover $(\overline{\mathcal{S}}_{g-1,2}^{--})'$ of $\overline{\mathcal{S}}_{g-1,2}^{--}$ and so Δ_0^{bb} is also irreducible. In order to conclude that $\Delta_0^{bn}, \Delta_0^{nb}$ and Δ_0^{nn} are also irreducible we argue in a similar way, but we need the irreducibility of different spaces; see Corollary 4.27.

1.8 Ramification

Our goal in this section is to compute the canonical class of $\overline{\mathcal{S}}_g^{--}$ and of its coarse moduli space $\overline{S}_g^{--}$ in terms of the standard divisor classes. The way to do it is to compute the ramification over $\overline{\mathcal{S}}_g^{-}$ and apply the Riemann–Hurwitz formula. We assume throughout $g \geq 3$.

The standard divisor classes are the boundary classes $\delta_i^{xy} = \mathcal{O}_{\overline{\mathcal{S}}_g^{--}}(\Delta_i^{xy})$ and the Hodge class λ defined as usual. The boundary classes of $\overline{\mathcal{S}}_g^{-}$ will be denoted by δ_i^{+} and the boundary classes of $\overline{\mathcal{M}}_g$ will be denoted by δ_i. The construction of Hodge class respects base change so we may view it as being pulled back from $\overline{\mathcal{M}}_g$ to the other spaces and we always denote it by λ.

1.8.1 Ramification between stacks

Let $f_1 : \overline{\mathcal{S}}_g^{--} \to \overline{\mathcal{S}}_g^{-}$ be the map keeping the first spin structure and forgetting the second one. Similarly define $f_2 : \overline{\mathcal{S}}_g^{--} \to \overline{\mathcal{S}}_g^{-}$ as the map keeping the second spin structure.

Lemma 1.50. *The ramification divisor class of f_1 is δ_0^{nb} and of f_2 is δ_0^{bn}.*

Proof. We will prove this for f_1, the proof for f_2 being entirely symmetric.

Pick a stable spin curve $\xi := (X, (\mathcal{E}_i, \alpha_i)_{i=1}^2, \varphi)$ where φ is the synchronization data. We label the nodes of X by $x_1, \ldots, x_n$. The universal deformation functor of X is pro-represented by $T := k[[t_1, \ldots, t_{3g-3}]]$ where we index the first n parameters so that t_i cuts out the locus where the node x_i persists. There are integers $0 \leq n' \leq n'' \leq n$ such that, after re-indexing the nodes if necessary, the spin structure $\mathcal{E}_1$ is singular only on the nodes $x_1, \ldots, x_{n'}$ and either $\mathcal{E}_1$ or $\mathcal{E}_2$ is singular at each of the nodes $x_1, \ldots, x_{n''}$.

By Corollary I.4.22 we can describe the local deformation functors of the spin curves and the maps between them as follows. The local deformation functor of ξ is pro-represented by $T'' := k[[v_1, \ldots, v_{3g-3}]]$ and that of $f_1(\xi)$ is pro-represented by $T' := k[[u_1, \ldots, u_{3g-3}]]$ where we chose the parameters so that the maps $a : T \to T'$ and $b : T \to T''$ satisfy:

$$a(t_i) = \begin{cases} u_i^2 & : \quad i \leq n' \\ u_i & : \quad i > n' \end{cases} \qquad b(t_i) = \begin{cases} v_i^2 & : \quad i \leq n'' \\ v_i & : \quad i > n'' \end{cases}$$

We may further arrange it so that the map $c : T' \to T''$ corresponding to the forgetful functor f_1 maps $u_i \mapsto v_i$ for all i. Therefore ξ is a point of ramification of f_1 iff $n' \neq n''$. In particular, f_1 is not ramified over smooth spin curves. To describe the ramification divisor of f_1 we need only consider curves with a single node, i.e., when $n = 1$. Then ramification occurs iff $n' = 0$ and $n'' = 1$. That is, the second spin structure is singular on the node but the first spin structure is smooth. This happens only on Δ_0^{nb}. Clearly, the order of ramification is 2. □

Let $\rho : \overline{\mathcal{S}}_g^- \to \overline{\mathcal{M}}_g$ be the map forgetting the spin structure. As the local nature of this map has been studied in [Cor89] we will only record its ramification locus here. However, the argument above applies, essentially verbatim, to this setting.

Lemma 1.51 (Cornalba)**.** *The morphism ρ has ramification divisor $\delta_0^b + \sum_{i=1}^{g-1} \delta_i^+$.*

1.8.2 Ramification over coarse moduli

Let $\overline{S}_g^{--}$ be the coarse moduli scheme corresponding to $\overline{\mathcal{S}}_g^{--}$ and let $\mathfrak{m} : \overline{\mathcal{S}}_g^{--} \to \overline{S}_g^{--}$ denote the coarse moduli map. Given a double spin curve $\xi \in \overline{\mathcal{S}}_g^{--}$ we will denote its image in $\overline{S}_g^{--}$ by $[\xi]$. Similarly we define $\overline{S}_g^-$ and $\overline{M}_g$ to be the coarse moduli schemes corresponding to $\overline{\mathcal{S}}_g^-$ and $\overline{\mathcal{M}}_g$ respectively. In all three cases we will use $\mathfrak{m}$ to denote the coarse moduli map.

Notation 1.52. Let $\tilde{\delta}_i = \delta_i^{++} + \delta_i^{+-} + \delta_i^{+=} + \delta_{g-1}^{-=}$.

Lemma 1.53. *The morphism $\mathfrak{m} : \overline{\mathcal{S}}_g^{--} \to \overline{S}_g^{--}$ has ramification divisor $\sum_{i=2}^{g-2} \tilde{\delta}_i + 3(\tilde{\delta}_1 + \tilde{\delta}_{g-1})$*

Proof. Let $\xi = (X, (\mathcal{E}_1, \mathcal{E}_2)_{i=1}^2, \varphi)$ be a double spin curve and let $\xi' := f_1(\xi)$. Let $\mathrm{Aut}(\xi), \mathrm{Aut}(\xi')$ and $\mathrm{Aut}(X)$ be the automorphism groups of the respective double spin, spin and ordinary curves. We define the formal power series rings T, T', T'' and the maps a, b, c between them as in the proof of Lemma 1.50. Then T'' is the formal local ring at ξ of $\overline{\mathcal{S}}_g^{--}$ and the ring of invariants $(T'')^{\mathrm{Aut}(\xi)} \hookrightarrow T''$ is isomorphic to $\hat{\mathcal{O}}_{\overline{S}_g^{--},[\xi]}$ with the standard inclusion map corresponding to the moduli map $\hat{\mathcal{O}}_{\overline{S}_g^{--},[\xi]} \hookrightarrow \hat{\mathcal{O}}_{\overline{\mathcal{S}}_g^{--},\xi}$.

Let R stand for the ramification divisor of m. If X is smooth, outside of a codimension ≥ 2 locus, $\mathrm{Aut}(X) = 1$. Furthermore, the only automorphisms of the spin structures are the multiplication by ± 1, which do not act on the pro-representing ring as we discussed in Section I.4.5. Thus R is supported in the boundary. For the rest of this proof, we may assume that X has only a single node.

In order to avoid elliptic tail automorphisms for now, let us assume $X \in \Delta_i^{xy}$ where $i \neq 1, g-1$, In particular, with X sufficiently general in boundary we may assume $\mathrm{Aut}(X) = 1$. Therefore, the only automorphisms of the double spin curve ξ are the *inessential* automorphisms. See Definition I.4.24 and Theorem I.4.39 for an explicit description of the action on the pro-representing rings. We recall the results here.

If X is irreducible, then $\underline{\mathrm{Aut}}_0(\xi) = 1$ so $(T'')^{\underline{\mathrm{Aut}}_0(\xi)} = T''$ and m is not ramified. In particular, m is not ramified over $\delta_0^{nn}, \delta_0^{nb}, \delta_0^{bn}, \delta_0^{bb}, \delta_0^{b=}$.

If X is reducible, but has no elliptic tail automorphisms then $\underline{\mathrm{Aut}}_0(\xi) \simeq \mathbb{Z}/2\mathbb{Z}$ where the action of the generator is $T'' \to T'' : v_1 \mapsto -v_1$ while the other variables are fixed.

If X is reducible with elliptic tail (and a single node) we remark that the action of the inessential automorphisms of ξ and ξ' on the pro-representing rings $T' \simeq T''$ are identical. Therefore we may apply Theorem 2.12 and Remark 2.17 of [Lud10] to realize that $\underline{\mathrm{Aut}}(\xi) \simeq \mathbb{Z}/4\mathbb{Z}$ with the action of a generator given by $T'' \to T'' : v_1 \mapsto \sqrt{-1}v_1$ the other variables being fixed. $\square$

Arguing in the same way, we also obtain the following:

Lemma 1.54. *The ramification divisor of* $\mathfrak{m} : \overline{\mathcal{S}}_g^- \to \overline{S}_g^-$ *is* $\sum_{i=2}^{g-2} \delta_i^+ + 3(\delta_1^+ + \delta_{g-1}^+)$.

1.8.3 Pullback of boundary classes

For convenient use later in the thesis, we collect the pullbacks of boundary divisors all in one place. Let $\rho : \overline{\mathcal{S}}_g^- \to \overline{\mathcal{M}}_g$ the forgetful map. Then Proposition 7.2 of [Cor89] contains the following statement:

$$\rho^*\delta_i = \begin{cases} \delta_0^n + 2\delta_0^b & : \quad i = 0 \\ 2(\delta_i^+ + \delta_{g-i}^+) & : \quad i \neq 0, \frac{g}{2} \\ 2\delta_{\frac{g}{2}}^+ & : \quad g = 2i \end{cases} \tag{1.8.2}$$

Recall that for $j = 1, 2$ the map $f_j : \overline{\mathcal{S}}_g^{--} \to \overline{\mathcal{S}}_g^-$ keeps the j-th spin structure and forgets the other one. Combining Equation 1.8.2 with our computation of the ramification locus of f_j we get the following:

$$\begin{aligned} f_1^*\lambda &= \lambda & f_2^*\lambda &= \lambda \\ f_1^*\delta_0^b &= \delta_0^{bn} + \delta_0^{bb} + \delta_0^{b=} & f_2^*\delta_0^b &= \delta_0^{nb} + \delta_0^{bb} + \delta_0^{b=} \\ f_1^*\delta_0^n &= \delta_0^{nn} + 2\delta_0^{nb} & f_2^*\delta_0^n &= \delta_0^{nn} + 2\delta_0^{bn} \\ f_1^*\delta_i^+ &= \delta_i^{++} + \delta_i^{+-} + \delta_i^{+=} + \delta_i^{-=} & f_2^*\delta_i^+ &= \delta_i^{++} + \delta_i^{-+} + \delta_i^{+=} + \delta_i^{-=} \end{aligned} \tag{1.8.3}$$

Let $\pi : \overline{\mathcal{S}}_g^{--} \to \overline{\mathcal{M}}_g$ be the map forgetting both spin structures. Combining Equations 1.8.2 and 1.8.3 we get:

$$\pi^*\delta_i = \begin{cases} \delta_0^{nn} + 2(\delta_0^{nb} + \delta_0^{bn} + \delta_0^{bb} + \delta_0^{b=}) & : \quad i = 0 \\ 2(\delta_i^{++} + \delta_i^{+-} + \delta_i^{+=} + \delta_i^{-=} + \delta_{g-i}^{++} + \delta_{g-i}^{+-} + \delta_{g-i}^{+=} + \delta_{g-i}^{-=}) & : \quad i \neq 0, \frac{g}{2} \\ 2(\delta_{\frac{g}{2}}^{++} + \delta_{\frac{g}{2}}^{+-} + \delta_{\frac{g}{2}}^{+=} + \delta_{\frac{g}{2}}^{-=}) & : \quad g = 2i \end{cases} \tag{1.8.4}$$

1.8.4 The canonical classes

Recall [HM82] that the canonical class of $\overline{\mathcal{M}}_g$ is given by $\omega_{\overline{\mathcal{M}}_g} \equiv 13\lambda - 2\sum_{i=1}^{\lfloor \frac{g}{2} \rfloor} \delta_i$. By applying Riemann–Hurwitz formula to the maps studied so far, we can obtain the other canonical classes. Note that the results for $\overline{\mathcal{S}}_g^-$ are known, see [FV14].

Notation 1.55. For a given map f, let $\operatorname{ram}(f)$ stand for its ramification divisor.

Lemma 1.56. *In* $\operatorname{Pic}_{\mathbb{Q}}(\overline{\mathcal{S}}_g^-)$*, we have:*

$$\omega_{\overline{\mathcal{S}}_g^-} \equiv 13\lambda - 2\delta_0^n - 3\delta_0^b - 3\sum_{i=1}^{g-1}\delta_i^+.$$

Proof. With $\rho : \overline{\mathcal{S}}_g^- \to \overline{\mathcal{M}}_g$ for forgetful map, Riemann–Hurwitz implies $\rho^*\omega_{\overline{\mathcal{M}}_g} \equiv \omega_{\overline{\mathcal{S}}_g^-} + \operatorname{ram}(\rho)$. We computed $\operatorname{ram}(\rho)$ in Lemma 1.51 and the formulae for the pullback ρ^* are given in Equation 1.8.2, putting these together we get the desired result. □

Recall $\mathfrak{m} : \overline{\mathcal{S}}_g^- \to \overline{S}_g^-$ is the coarse moduli map. The pullback map $\mathfrak{m}^* : \operatorname{Pic}_{\mathbb{Q}}(\overline{S}_g^-) \to \operatorname{Pic}_{\mathbb{Q}}(\overline{\mathcal{S}}_g^-)$ is an isomorphism since $\overline{S}_g^-$ has only finite quotient singularities. We let $K_{\overline{S}_g^-}$ denote the canonical class of $\overline{S}_g^-$.

Corollary 1.57. *In* $\operatorname{Pic}_{\mathbb{Q}}(\overline{\mathcal{S}}_g^-)$ *we have:*

$$\mathfrak{m}^*K_{\overline{S}_g^-} \equiv 13\lambda - 2\delta_0^n - 3\delta_0^b - 4\sum_{i=2}^{g-2}\delta_i^+ - 6(\delta_1^+ + \delta_{g-1}^+)$$

Proof. Applying Riemann–Hurwitz to $\mathfrak{m}$ we get $\omega_{\overline{\mathcal{S}}_g^-} \equiv \mathfrak{m}^*K_{\overline{S}_g^-} + \operatorname{ram}(\mathfrak{m})$. Now substitute Lemma 1.54. □

Remark 1.58. The formula above looks very different from the one given in [FV14], so we restate it in the following form. In $\mathrm{CH}^1_{\mathbb{Q}}(\overline{S}_g^-)$ we have:

$$K_{\overline{S}_g^-} \equiv 13\lambda - 2[\Delta_0^n] - 3[\Delta_0^b] - 2\sum_{i=2}^{g-2}[\Delta_i^+] - 3(\frac{[\Delta_1^+]}{2} + \frac{[\Delta_{g-1}^+]}{2}),$$

where $[\Delta_i^x]$ represents the class of the boundary divisor of the *coarse* moduli space $\overline{S}_g^-$. See Section 1.8.2 above. Now, this formula and the one in [FV14] agree. The classes $[\Delta_1^+]$ and $[\Delta_{g-1}^+]$ in *loc. cit.* are implicitly divided by 2 to account for the elliptic tail automorphisms, which we made explicit here.

Proposition 1.59. *In* $\operatorname{Pic}_{\mathbb{Q}}(\overline{\mathcal{S}}_g^{--})$*, we have:*

$$\omega_{\overline{\mathcal{S}}_g^{--}} \equiv 13\lambda - 2\delta_0^{nn} - 3\delta_0^{nb} - 3\delta_0^{bn} - 3\delta_0^{bb} - 3\delta_0^{b=} - 3\sum_{i=1}^{g-1}(\delta_i^{++} + \delta_i^{+=} + \delta_i^{+-} + \delta_{g-1}^{-=}).$$

Proof. Let $f_1 : \overline{\mathcal{S}}_g^{--} \to \overline{\mathcal{S}}_g^-$ be the map forgetting the second spin structure and retaining the first. The ramification divisor of f_1 is δ_0^{nb} as we proved in Lemma 1.50. So the Riemann–Hurwitz formula gives us:

$$\omega_{\overline{\mathcal{S}}_g^{--}} \equiv f_1^*(\omega_{\overline{\mathcal{S}}_g^-}) + \delta_0^{nb}.$$

Now substituting the expression for $\omega_{\overline{\mathcal{S}}_g^-}$ from Lemma 1.56 and the pullback of divisors from Lemma 1.8.3 we get the stated result. □

Now let $\mathfrak{m} : \overline{\mathcal{S}}_g^{--} \to \overline{S}_g^{--}$ be the coarse moduli map. Recall that $\mathfrak{m}^* : \mathrm{Pic}_{\mathbb{Q}}(\overline{S}_g^{--}) \to \mathrm{Pic}_{\mathbb{Q}}(\overline{\mathcal{S}}_g^{--})$ is an isomorphism because $\overline{S}_g^{--}$ has only finite quotient singularities.

Notation 1.60. Let $\tilde{\delta}_i = \delta_i^{++} + \delta_i^{+-} + \delta_i^{+=} + \delta_{g-i}^{-=}$.

Corollary 1.61. *The canonical class of the coarse moduli space $\overline{S}_g^{--}$ in $\mathrm{Pic}_{\mathbb{Q}}(\overline{\mathcal{S}}_g^{--})$ is of the following form:*

$$\mathfrak{m}^* K_{\overline{S}_g^{--}} \equiv 13\lambda - 2\delta_0^{nn} - 3\delta_0^{nb} - 3\delta_0^{bn} - 3\delta_0^{bb} - 3\delta_0^{b=} - 4\sum_{i=2}^{g-2} \tilde{\delta}_i - 6(\tilde{\delta}_1 + \tilde{\delta}_{g-1}).$$

Proof. Once again we apply Riemann–Hurwitz to the formula given in Proposition 1.59 and using Lemma 1.53 for the ramification divisor. □

1.9 There are no other boundary components

Here we will give a proof of Proposition 1.45 that does not rely on exhaustion. We will work with the forgetful map $\pi : \overline{\mathcal{S}}_g^{--} \to \overline{\mathcal{M}}_g$ between the stacks and assume $g \geq 3$ throughout. As we will use them repeatedly, let us begin by introducing the notations $n^-(g)$ and $n^+(g)$.

Notation 1.62. On a smooth curve of genus g there are $n^-(g) := \binom{2^g}{2}$ odd theta characteristics and $n^+(g) := \binom{2^g+1}{2}$ even theta characteristics, see Mumford [Mum71].

In computing the degree $\deg \pi$ we must account for the global automorphisms of $\overline{\mathcal{S}}_g^{--}$ which are obtained by scaling each of the odd theta characteristics by ± 1. However, if we systematically ignore this factor of 4 we will still get correct equalities at the end. Moreover, the arguments will be more transparent and so we will ignore this factor. With this agreement, we have $\deg \pi = n^-(g)(n^-(g) - 1)$ since, over a smooth curve $C \in \mathcal{M}_g$ the fiber $\pi^{-1}(C)$ is in bijection with ordered pairs of distinct odd theta characteristics on C.

Let d_i^{xy} denote the degree of Δ_i^{xy} over its image Δ_i in $\overline{\mathcal{M}}_g$. In particular, $\pi_* \delta_i^{xy} = d_i^{xy} \delta_i$. Similarly, $\pi_* \pi^* \delta_i = (\deg \pi)\delta_i$ which when applies to Equation 1.8.4 gives us the following formulae:

$$\deg \pi = \begin{cases} d_0^{nn} + 2(d_0^{bb} + d_0^{b=} + d_0^{bn} + d_0^{nb}) & : \quad i = 0 \\ 2(d_i^{++} + d_i^{+-} + d_i^{-+} + d_i^{--} + d_i^{+=} + d_i^{-=} + d_{g-i}^{+=} + d_{g-i}^{-=}) & : \quad i \neq 0, \frac{g}{2} \\ 2(d_{\frac{g}{2}}^{++} + d_{\frac{g}{2}}^{+-} + d_{\frac{g}{2}}^{+=} + d_{\frac{g}{2}}^{-=}) & : \quad g = 2i \end{cases} \tag{1.9.5}$$

Using the examples given in Section 1.6 we can compute each of the degrees d_i^{xy}. However, double spin curve of type $i > 0$ will have extra automorphisms that do not appear over a smooth double spin curve; the inessential automorphisms (see Definition I.4.24). This means that each double spin curve over Δ_i must be counted with multiplicity $\frac{1}{2}$ when $i \neq 0$. There are also the elliptic tail automorphisms when $i = 1, g-1$ but these automorphisms exist also on the base Δ_1 and will not impact the degree.

Notation 1.63. Let N_i^{xy} be the degree of Δ_i^{xy} over Δ_i considered as *coarse moduli spaces.* So we have:

$$N_i^{xy} = \begin{cases} 2d_i^{xy} & : \quad i \neq 0, 1, g-1 \\ d_i^{xy} & : \quad i = 0 \end{cases}$$

For instance, N_0^{bb} is the number of isomorphism classes of double spin structures as in Example 1.26, defined over a general irreducible nodal curve. Given such $X \in \Delta_0 \subset \overline{\mathcal{M}}_g$ with normalization $\nu : C \to X$ we simply have to fix a pair of distinct odd theta characteristics on the genus $g-1$ curve C *and* fix one of the two possible synchronizations. Therefore $N_0^{bb} = 2n^-(g-1)(n^-(g-1)-1)$.

To compute N_0^{nn} we observe that we need to pick two *distinct* roots of $\omega_C(p+q)$ and then for each root there is a unique gluing that must be performed in order to obtain an odd theta characteristics on X. Thus $N_0^{nn} = 2^{2g-2}(2^{2g-2}-1)$.

To compute N_i^{++} one starts with a general nodal curve $X = C_1 \cup_{p\sim q} C_2 \in \Delta_i$, $g(C_1) = i$, and then picks a pair of *distinct* even theta characteristics on C_1 and a pair of *distinct* odd theta characteristics on C_2. The spin structures where the even theta characteristics on C_1 coincide is counted by $N_i^{+=}$ and the spin structures where the odd theta characteristics on C_2 coincide is counted by $N_{g-i}^{-=}$.

The other values are computed in a similar way, so we state them without proof:

$$\begin{aligned}
N_0^{bb} &= 2n^-(g-1)(n^-(g-1)-1) \\
N_0^{b=} &= n^-(g-1) \\
N_0^{nb} &= N_0^{bn} = 2^{2g-2}\, n^-(g-1) \\
N_0^{nn} &= 2^{2g-2}(2^{2g-2}-1) \\
N_i^{++} &= n^+(i)(n^+(i)-1)n^-(g-i)(n^-(g-i)-1) \\
N_i^{+-} &= N_i^{-+} = n^+(i)n^-(i)n^-(g-i)n^+(g-i) \\
N_i^{--} &= N_{g-i}^{++} \\
N_i^{+=} &= n^+(i)n^-(g-i)(n^-(g-i)-1) \\
N_i^{-=} &= n^-(i)n^+(g-i)(n^+(g-i)-1)
\end{aligned}$$

Whereas for $i = \frac{g}{2}$ we have the following equalities:

$$\begin{aligned}
N_{\frac{g}{2}}^{++} &= 2n^+(\frac{g}{2})(n^+(\frac{g}{2})-1)n^-(\frac{g}{2})(n^-(\frac{g}{2})-1) \\
N_{\frac{g}{2}}^{+-} &= 2n^+(\frac{g}{2})^2 n^-(\frac{g}{2})^2 \\
N_{\frac{g}{2}}^{+=} &= 2n^+(\frac{g}{2})n^-(\frac{g}{2})(n^-(\frac{g}{2})-1) \\
N_{\frac{g}{2}}^{-=} &= 2n^-(\frac{g}{2})n^+(\frac{g}{2})(n^+(\frac{g}{2})-1)
\end{aligned}$$

The identities in Equation 1.9.5 can now be checked by any computer algebra system, or by hand.

Chapter 2

Locus of contact points

Due to the nature of the problem we are interested in, rather than the full moduli space $\overline{\mathcal{S}}_g^{--}$ we will work only with odd theta characteristics that have the minimal number of sections, i.e., with $h^0 = 1$. As we will have to make this distinction throughout the text, we give it a name and introduce a notation for the corresponding substack.

Definition 2.1. Let (X, η) be a possibly singular spin curve. If $h^0(\eta)$ is minimal, that is $h^0(\eta) \in \{0, 1\}$, then we will call η a *rigid* theta characteristic and the pair (X, η) will be a *rigid spin curve*. Similarly, we will say a double spin curve (X, η_1, η_2) is rigid if $h^0(\eta_i)$ are both minimal.

Definition 2.2. If (X, η) is a rigid odd spin curve then any point $p \in X$ such that $p \leq \eta$ will be called a *contact point* of (X, η).

Definition 2.3. Let $\mathcal{U}_g^- \subset \mathcal{S}_g^-$ be the open substack of $\mathcal{S}_g^-$ consisting only of rigid spin curves. Similarly define $\mathcal{U}_g^{--} \subset \mathcal{S}_g^{--}$. In order to define $\overline{\mathcal{U}}_g^- \subset \overline{\mathcal{S}}_g^-$ and $\overline{\mathcal{U}}_g^{--} \subset \overline{\mathcal{S}}_g^{--}$ we demand that the spin curves be rigid, but we require an additional generality condition on the boundary curves which is to be made precise in Definition 2.14.

Remark 2.4. It is well known that the complement of $\mathcal{U}_g^-$ in $\mathcal{S}_g^-$ is of codimension 3, see [Har82]. The locus of theta characteristics that are not rigid comprise a codimension 1 locus in $\mathcal{S}_g^+$. Using this one can see that the complement of $\overline{\mathcal{U}}_g^-$ in $\overline{\mathcal{S}}_g^-$ will have components that are of codimension 2 lying entirely in the boundary. However, $\overline{\mathcal{S}}_g^- \setminus \overline{\mathcal{U}}_g^-$ will not contain any components of codimension 1. Similarly, the complement of $\overline{\mathcal{U}}_g^{--}$ in $\overline{\mathcal{S}}_g^{--}$ contains components of codim ≥ 2. This implies $\mathrm{Pic}(\overline{\mathcal{S}}_g^{--}) \simeq \mathrm{Pic}(\overline{\mathcal{U}}_g^{--})$.

2.1 Contact locus for smooth curves

Let $B \to \mathcal{U}_g^{--}$ correspond to a family of odd spin curves over B, say $(\pi : \mathcal{X} \to B, L, \alpha : L^{\otimes 2} \to \omega_{\mathcal{X}/B})$. There is a relative Cartier divisor on $\mathcal{X} \to B$, which we may denote by $\mathcal{D}$, restricting on each fiber $\mathcal{X}_b$ to the unique divisor in the linear system $|L_b|$. We will now describe the line bundle $\mathcal{O}_{\mathcal{X}}(\mathcal{D})$ in terms of L.

2.1.1 Idea for the construction

Let us begin by an observation on a single odd spin curve $(C, \eta) \in \mathcal{U}_g^{--}$. Since $h^0(\eta) = 1$ the unique divisor D_η in $|\eta|$ may be viewed as the base locus of η. In other words, the natural restriction map of global sections fit into the following exact sequence:

$$0 \longrightarrow \mathrm{H}^0(C, \eta) \otimes \mathcal{O}_C \longrightarrow \eta \longrightarrow \mathcal{O}_{D_\eta} \longrightarrow 0 \tag{2.1.1}$$

2.1.2 The construction on universal curve

This construction works for families as well. Consider the universal curve $\pi : \mathcal{C} \to \mathcal{U}_g^{--}$ with the spin structure $(\mathcal{L}, \mathcal{L}^{\otimes 2} \xrightarrow{\sim} \omega_\pi)$. Because the base is a reduced algebraic stack the pushforwards $\pi_*\mathcal{L}$ and $\mathrm{R}^1\pi_*\mathcal{L}$ are both locally free of rank 1 and they satisfy base change. Adjunction provides us with the canonical map $\pi^*\pi_*\mathcal{L} \to \mathcal{L}$ and as $\mathcal{C}$ is integral this morphism of line bundles must be an inclusion. This defines a quotient Q:

$$0 \longrightarrow \pi^*\pi_*\mathcal{L} \longrightarrow \mathcal{L} \longrightarrow Q \longrightarrow 0$$

As $\pi_*\mathcal{L}$ satisfies base change, restriction of this exact sequence to closed fibers recovers the exact sequence 2.1.1. Thus Q is supported on exactly the relative contact locus $\mathcal{D}$ of $\mathcal{C} \to \mathcal{U}_g^{--}$. The ideal sheaf of $\mathcal{D}$ is then $\pi^*\pi_*\mathcal{L} \otimes \mathcal{L}^\vee \hookrightarrow \mathcal{O}_\mathcal{C}$ and thus $\mathcal{O}(\mathcal{D}) \simeq \mathcal{L} \otimes (\pi^*\pi_*\mathcal{L})^\vee$. This proves the following.

Proposition 2.5. *The contact locus $\mathcal{D} \subset \mathcal{C} \to \overline{\mathcal{U}}_g^{--}$ is a relative Cartier divisor and it satisfies:*

$$\mathcal{O}_\mathcal{C}(\mathcal{D}) \simeq \mathcal{L} \otimes \pi^*(\pi_*\mathcal{L})^\vee.$$

Going back to our family of odd spin curves $(\pi : \mathcal{X} \to B, L, \alpha)$, the line bundle L is the pullback of $\mathcal{L}$ to $\mathcal{X}$. Therefore we have the following result:

Corollary 2.6. *The relative contact points $\mathcal{D}_\mathcal{X}$ of $(\pi : \mathcal{X} \to B, L, \alpha)$ satisfies $\mathcal{O}_\mathcal{X}(\mathcal{D}_\mathcal{X}) \simeq L \otimes (\pi^*\pi_*L)^\vee$.*

Proof. Since $\pi_*\mathcal{L}$ satisfies base change, the expression in Proposition 2.5 pullsback to the stated expression. □

2.2 Limit contact points

In the previous section we described the contact points of odd theta characteristics of smooth curves. We now wish to extend this to stable curves that are general in their respective boundary component.

A generic element of Δ_i^+ is of the form (X, L) where $X = C_1 \cup_p \mathbb{P}^1 \cup_q C_2$, $g(C_1) = i$ and $L = (\eta_1, \mathcal{O}(1), \eta_2)$, where $L|_{C_j} = \eta_j$. We omit the gluing data from these expressions. We assumed X generic, so $h^0(\eta_1) = 0$ and $h^0(\eta_2) = 1$. In particular, the base locus of L will be the union of the base locus of η_2 and the entire component C_1.

If we take a 1-parameter family of smooth spin curves whose relative contact locus is defined, and we take as a special fiber (X, L) then we can ask for the limit of the relative contact locus on X. This is answered by a limit linear series argument.

Definition 2.7. If by this construction (X, L) identifies a divisor D on X that is independent of the limiting family, then D will be called a *limit contact divisor* on X and any point $x \leq D$ will be called a *limit contact point.* We will often drop the word "limit" and refer to them as contact divisors.

Lemma 2.8. *Let $(X, (\eta_1, \mathcal{O}(1), \eta_2)) \in \Delta_i^+$ be general only in the sense that $h^0(\eta_1) = 0$, $h^0(\eta_2) = 1$ and $p \not\leq \eta_2$. Then the limit contact locus on X is the base locus of $(\eta_1 + p, \mathcal{O}_{\mathbb{P}^1}, \eta_2)$.*

Proof. By a limit linear series argument, we know that the limit contact locus must be linearly equivalent to $(\eta_1 + ap, \mathcal{O}(1 - a - b), \eta_2 + bq)$ where $a, b \in \mathbb{Z}$. This line bundle is effective iff $1 \leq a$, $a + b \leq 1$ and $0 \leq b$ since $p \not\leq \eta_2$. There is a unique solution to this set of inequalities, namely $a = 1, b = 0$. □

Remark 2.9. We emphasize that the even theta characteristic, after the twist will have one dimensional sections: $h^0(\eta_1 + p) = 1$. Therefore, we are claiming that the limit contact locus will be supported only on C_1 and C_2 represented by the divisors $\eta_1 + p$ on C_1 and η_2 on C_2.

Take an element $(X, \eta) \in \Delta_0^n$ such that $X = C/(p \sim q)$ for a smooth marked curve $(C, p, q) \in \mathcal{M}_{g-1,2}$. We show in Lemma 3.30 that, assuming (C, p, q) is general, any root of $\sqrt{\omega_C(p + q)}$ will have one dimensional global sections. Therefore, the following lemma is applicable in general. We omit its proof as it is immediate.

Lemma 2.10. *If $(X, \eta) \in \Delta_0^n$ is as above with $h^0(\eta|_C) = 1$ then the limit contact locus of (X, η) will simply be the base locus of η.*

Finally consider an element $(X, L) \in \Delta_0^b$ such that

$$X = C \cup_{\substack{p \sim 0 \\ q \sim \infty}} \mathbb{P}^1.$$

Then $\eta := L|_C$ is an odd theta characteristic on C and $L|_{\mathbb{P}^1} \simeq \mathcal{O}_{\mathbb{P}^1}(1)$. We will assume that $h^0(\eta) = 1$. Arguing as in Lemma 2.8 we then immediately obtain the following result.

Lemma 2.11. *If $p + q \not\leq \eta$ then the limit contact locus on (X, L) is the base locus of L which is the base divisor of η and a point on $\mathbb{P}^1$.*

Remark 2.12. We will have more to say about the position of this point on $\mathbb{P}^1$ in Section 3.2.

Definition 2.13. Let us say a rigid spin curve $(X, \eta) \in \overline{\mathcal{S}}_g^-$ is *contact general* if the following conditions are satisfied:

- X has at most a single node,
- If for some $i = 1, \dots, g-1$ we have $X \in \Delta_i^+$, that is $X = C_1 \cup_{p\sim q} C_2$ with $g(C_1) = i$ and $\eta = (\eta_1, \eta_2)$, then $p \not\leq \eta_2$.
- If $X \in \Delta_0^b$, that is $X = C/(p \sim q)$, then $p + q \not\leq \eta|_C$.
- If $X \in \Delta_0^n$, $X = C/(p \sim q)$ then $h^0(\eta|_C) = 1$.

Definition 2.14. We may now complete the definition of $\overline{\mathcal{U}}_g^-$ and $\overline{\mathcal{U}}_g^{--}$ by insisting that they contain only contact general spin curves, which are also rigid by definition.

Lemma 2.15. *If (X, η) is a contact general curve and $D \subset X$ is the corresponding contact divisor then $h^0(\mathcal{O}_X(D)) = h^1(\mathcal{O}_X(D)) = 1$.*

Proof. If $(X, \eta) \in \Delta_0^n$ then $\eta^{\otimes 2} \simeq \omega_X$ so $h^0(\eta) = h^1(\eta)$ and the first of these values is 1 by definition of contact general. Now use $\mathcal{O}_X(D) \simeq \eta$.

If $(X, L) \in \Delta_0^b$ then $\eta = (\eta_C, \mathcal{O}_{\mathbb{P}^1}(1))$ with $h^0(\eta_C) = 1$. Then $h^1(X, \eta) = h^0(X, \omega_X \otimes \eta) = h^0(X, (\eta_C + p + q, \mathcal{O}_{\mathbb{P}^1}(-1))) = h^0(C, \eta_C) = 1$.

If $(X, L) \in \Delta_i^+$ then $L = (\eta_1 + p, \mathcal{O}_{\mathbb{P}^1}, \eta_2)$ with $h^0(L) = 1$. Then $h^1(L) = h^1(\omega_X \otimes L^\vee) = h^0(X, (\eta_1, \mathcal{O}_{\mathbb{P}^1}, \eta_2 + q)) = h^0(C_2, \eta_2 + q)$ and since we assumed $q \not\leq \eta_2$ by Riemann–Roch we conclude that $h^0(C_2, \eta_2 + q) = 1$. $\square$

2.3 Limit contact locus

As we have done in Section 2.1 we wish to describe the *locus* of limit contact points on a family of curves. The first thing to do is to incorporate the twists we need on singular fibers in to the family of spin structures to get rid of the vertical base loci of a spin structure.

The locus Δ_i^+ is the image of the gluing map from $\overline{\mathcal{S}}_i^+ \times \overline{\mathcal{S}}_{g-i}^-$. Let us denote the image of the universal genus i curve by C_i^+ as it has a positive spin structure. This image lies within the universal spin curve over the boundary $\mathcal{C}|_{\Delta_i^+}$. Moreover, since the universal curve $\mathcal{C}$ is a smooth DM stack, C_i^+ is a Cartier divisor in $\mathcal{C}$.

Definition 2.16. On $\mathcal{C}$ we will define $\tilde{\mathcal{L}} = \mathcal{L}(-\sum_{i=1}^{g-1} C_i^+)$ and call it the *twisted spin structure*. On an arbitrary family of odd spin curves, given by $B \to \overline{\mathcal{S}}_g^{--}$, the twisted spin structure is to be the pullback of $\tilde{\mathcal{L}}$ to the family over B.

Lemma 2.17. *The pushforwards $\pi_*\tilde{\mathcal{L}}$ and $\mathrm{R}^1\pi_*\tilde{\mathcal{L}}$ are line bundles and in particular they satisfy base change.*

Proof. Since $\overline{\mathcal{U}}_g^{--}$ is reduced we need only check H^0 and H^1 are constant on geometric fibers. This we proved in Lemma 2.15. $\square$

Remark 2.18. Note that the pullback of C_i^+ to an arbitrary family need not be a divisor. For instance, if $B = \operatorname{Spec} k \to \Delta_i^+$ then $C_i^+|_B$ will be a component of the curve over B. This is why we needed to construct the twisted spin structure on the universal curve and extend to arbitrary families by pullback.

From our discussion in Section 2.2 we expect $\mathcal{D}$ to be a relative Cartier divisor related to $\tilde{\mathcal{L}}$. This relationship is made precise with the following result.

Proposition 2.19. *The limit contact locus $\mathcal{D} \subset \mathcal{C} \to \overline{\mathcal{U}}_g^{--}$ is a relative Cartier divisor and it satisfies:*

$$\mathcal{O}_{\mathcal{C}}(\mathcal{D}) \simeq \tilde{\mathcal{L}} \otimes \pi^*(\pi_*\tilde{\mathcal{L}})^\vee.$$

Proof. The argument is almost identical to that of Proposition 2.5 so we will be brief. By Lemma 2.17 the pushforward $\pi_*\tilde{\mathcal{L}}$ is a line bundle and thus the adjunction map $\pi^*\pi_*\tilde{\mathcal{L}} \to \tilde{\mathcal{L}}$ must be injective. It remains to show that the quotient $\tilde{\mathcal{L}}/\pi_*\pi^*\mathcal{L}$ is supported on $\mathcal{D}$. To see this, we pullback to closed fibers. If the closed fiber is smooth we recover the exact sequence 2.1.1. Otherwise our classification of limit contact divisors in Section 2.2 must be used. $\square$

Let $(\pi : \mathcal{X} \to B, L)$ be any family of spin curves such that the morphism $B \to \overline{\mathcal{S}}_g^{--}$ factors through $B \to \overline{\mathcal{U}}_g^{--}$. Denote by $\tilde{L}$ the twisted spin structure on $\mathcal{X} \to B$, i.e., the pullback of $\tilde{\mathcal{L}}$. Similarly, let $\mathcal{D}_{\mathcal{X}}$ be the pullback of the divisor $\mathcal{D}$ to $\mathcal{X}$. We then have:

Corollary 2.20. *The limit contact locus $\mathcal{D}_{\mathcal{X}} \subset \mathcal{X} \to B$ is a relative Cartier divisor and it satisfies $\mathcal{O}_{\mathcal{X}}(\mathcal{D}_{\mathcal{X}}) \simeq \tilde{L} \otimes \pi^*(\pi_*\tilde{L})^\vee$*

Proof. The only non-trivial statement here is that we may substitute $(\pi^*\pi_*\tilde{\mathcal{L}})|_{\mathcal{X}}$ with $\pi^*\pi_*\tilde{L}$. This follows from the fact that $\pi_*\tilde{\mathcal{L}}$ satisfies base change. $\square$

2.4 Common contact locus

We will now give a precise definition of the locus Ω_g, first introduced in the Preface, which is to be a closed substack of $\mathcal{U}_g^{--}$. We want the closed points of Ω_g to correspond to the set $\{(C, \eta_1, \eta_2) \mid \eta_1 \cap \eta_2 \neq \emptyset\}$ in $\mathcal{U}_g^{--}$. We will define $\overline{\Omega}_g$ to be the Zariski closure of Ω_g in $\overline{\mathcal{U}}_g^{--}$.

On the other hand, there is another natural compactification of Ω_g which we denote by $\widetilde{\Omega}_g \subset \overline{\mathcal{U}}_g^{--}$ whose closed points are to correspond to the set $\{(C, \eta_1, \eta_2) \mid D_1 \cap D_2 \neq \emptyset\}$ where D_i is the (limit) contact divisor of η_i. Because $\widetilde{\Omega}_g$ extends the moduli interpretation of Ω_g we will call it the *modular compactification* of Ω_g.

Consider the universal curve $\pi : \mathcal{C} \to \overline{\mathcal{U}}_g^{--}$ with the two spin structures $\mathcal{L}_1$ and $\mathcal{L}_2$. Each $\mathcal{L}_i$ defines its relative contact locus $\mathcal{D}_i \subset \mathcal{C}$. Let $\mathcal{C}^o \to \mathcal{U}_g^{--}$ denote the universal curve over $\mathcal{U}_g^{--}$, which is an open subspace of $\mathcal{C}$. Let $\mathcal{D}_i^0 \subset \mathcal{C}^o$ be the relative contact locus of $\mathcal{L}_i|_{\mathcal{U}_g^{--}}$.

Definition 2.21. Let $\Omega_g := \pi(\mathcal{D}_1^o \cap \mathcal{D}_2^o)$ and $\widetilde{\Omega}_g := \pi(\mathcal{D}_1 \cap \mathcal{D}_2)$. Let $\overline{\Omega}_g \subset \overline{\mathcal{U}}_g^{--}$ be the Zariski closure of Ω_g in $\overline{\mathcal{U}}_g^{--}$.

Remark 2.22. The construction of $\overline{\Omega}_g$ does *not* satisfy base change, because Zariski closure does not commute with base change. For instance, take a point in the boundary $\overline{\Omega}_g \setminus \Omega_g$ and base change to that point. Since Ω_g has empty intersection with that point, its closure will be empty.

The remark above may be trivial, but it is the source of much trouble. This remark, coupled with the lemma below is the reason why we introduce the alternative compactification $\widetilde{\Omega}_g$ and will go to great lengths to distinguish $\widetilde{\Omega}_g$ from $\overline{\Omega}_g$ in Chapter 3.

Lemma 2.23. *The construction of Ω_g, and of $\widetilde{\Omega}_g$, as the image of the intersection of two relative contact loci satisfies base change.*

Proof. This follows from Proposition 2.6 of [LK79]. We will present the argument here for $\widetilde{\Omega}_g$. In their notation we have $X = \mathcal{D}_1 \cap \mathcal{D}_2$, $Y = \overline{\mathcal{U}}_g^{--}$ and $S = \mathrm{Spec}(k)$. Flatness over S is immediate and the co-flatness of $X \to Y$ follows from Example 2.12.(2) of [LK79] so we may apply the cited proposition. $\square$

Remark 2.24. The moduli interpretation of Ω_g is as follows. Let $(\pi : \mathcal{X} \to B, L_1, L_2) \in \mathcal{U}_g^{--}$ be a family of double spin curves with relative contact loci $D_i \subset \mathcal{X}$. Then $(\mathcal{X} \to B, L_1, L_2) \in \Omega_g$ iff $\pi(D_1 \cap D_2) = B$. This is exactly what one would expect for reduced B. For non-reduced B this brings in the 'tangential' directions of the intersection $\mathcal{D}_1 \cap \mathcal{D}_2$ into consideration. The moduli interpretation of $\widetilde{\Omega}_g$ is almost identical except defined for families in $\overline{\mathcal{U}}_g^{--}$.

Lemma 2.25. *The spaces Ω_g, $\overline{\Omega}_g$ and $\widetilde{\Omega}_g$ are all pure of codimension 1.*

Proof. Since $\widetilde{\Omega}_g$ contains the other two, it will be sufficient to prove this statement for $\widetilde{\Omega}_g$. The relative Cartier divisors $\mathcal{D}_i$ can only intersect in components of codimension 1 or 2. However, since $\mathcal{D}_i$ contains no fibers of the universal curve (essentially by design) neither will $\mathcal{D}_1 \cap \mathcal{D}_2$. Since, $\overline{\mathcal{U}}_g^{--}$ is irreducible by Remark 4.24, if $\mathcal{D}_1 \cap \mathcal{D}_2$ had any codimension 1 components then it would dominate the base $\overline{\mathcal{U}}_g^{--}$. On the other hand, specializing to rigid odd hyperelliptic spin curves, we see that not all elements in $\overline{\mathcal{U}}_g^{--}$ have a common contact point. Since the construction of $\widetilde{\Omega}_g$ satisfies base change, $\widetilde{\Omega}_g$ can not be the entirety of $\overline{\mathcal{U}}_g^{--}$. This proves that $\mathcal{D}_1 \cap \mathcal{D}_2$ can not have any codimension 1 components. Then this intersection is pure of codimension 2 and quasi-finite over $\overline{\mathcal{U}}_g^{--}$. Thus $\widetilde{\Omega}_g = \pi(\mathcal{D}_1 \cap \mathcal{D}_2)$ is pure of codimension 1. $\square$

For future reference we now give an explicit definition of common contact and limit common contact curves, first introduced in the Preface.

Definition 2.26. Let C be a smooth curve over k. If $C \in \mathcal{M}_g$ is contained in the image of Ω_g then we will call C a *common contact curve.* If $(C, \eta_1, \eta_2) \in \Omega_g$ then we will call (η_1, η_2) a *common contact pair.* If a stable curve $C \in \overline{\mathcal{M}}_g$ lies in the image of $\overline{\Omega}_g$ then C will be called a *limit contact curve.*

2.5 Divisor class computation

Let $\mathcal{D}_1, \mathcal{D}_2 \subset \mathcal{C} \to \overline{\mathcal{U}}_g^{--}$ be the two contact loci introduced in Section 2.4. As the divisor classes $[\overline{\Omega}_g]$ and $[\widetilde{\Omega}_g] = [\pi(\mathcal{D}_1 \cap \mathcal{D}_2)]$ are closely related to $\pi_*([\mathcal{D}_1] \cdot [\mathcal{D}_2])$ our primary goal in this section will be to compute $\pi_*([\mathcal{D}_1]\cdot[\mathcal{D}_2])$ in $\mathrm{Pic}_{\mathbb{Q}}(\overline{\mathcal{U}}_g^{--})$ in terms of the standard classes. Recall $\mathrm{Pic}_{\mathbb{Q}}(\overline{\mathcal{U}}_g^{--}) \simeq \mathrm{Pic}_{\mathbb{Q}}(\overline{\mathcal{S}}_g^{--})$ and so we will identify these two spaces for convenience.

In Section 2.5.1 we will recall the basic properties of the Deligne pairing from §13.5 of [ACG11] and a minor lemma will be proven. In Section 2.5.2 we will expand and simplify the expression $\pi_*([\mathcal{D}_1]\cdot[\mathcal{D}_2])$. Finally, in Section 2.5.3 we will put these parts together to compute $\pi_*([\mathcal{D}_1]\cdot[\mathcal{D}_2])$ and record the implications for the classes $[\widetilde{\Omega}_g]$ and $[\overline{\Omega}_g]$.

2.5.1 Preliminaries

Let $\pi : \mathcal{X} \to B$ be a family of stable curves and L_1, L_2 be a pair of line bundles on the total family $\mathcal{X}$. Then the Deligne pairing of L_1 and L_2, denoted $\langle L_1, L_2\rangle$, is a line bundle on the base B. The construction of the Deligne pairing respects base change and we have the following relation between the first Chern classes:

$$c_1\langle L_1, L_2\rangle = \pi_*(c_1(L_1)\cdot c_1(L_2)). \tag{2.5.2}$$

To simplify this expression further let us recall the relative determinant construction and a simple observation regarding it. Let $\pi : \mathcal{X} \to B$ be a family of stable curves of genus g and unless otherwise stated let B be irreducible.

Definition 2.27. For a vector bundle E of rank e on B let us denote by $\det E$ the line bundle $\bigwedge^e E$. For a coherent sheaf $\mathcal{G}$ on $\mathcal{X}$ denote by $\mathrm{d}_\pi(\mathcal{G})$ the relative determinant of $\mathcal{G}$ with respect to π, which is a line bundle on the base B. The construction of $\mathrm{d}_\pi(\mathcal{G})$ respects base change and when $\pi_*\mathcal{G}$ and $\mathrm{R}^1\pi_*\mathcal{G}$ are both locally free then $\mathrm{d}_\pi(\mathcal{G}) \simeq \pi_*\mathcal{G}\otimes(\mathrm{R}^1\pi_*\mathcal{G})^\vee$.

Remark 2.28. When the pushforwards of $\mathcal{G}$ are not locally free the following interpretation is often useful. For a base change $B' \to B$ let $\pi' : \mathcal{X}' \to B'$ be the pullback of π and $\mathcal{G}'$ the pullback of $\mathcal{G}$. It is possible to find an étale surjective $B' \to B$ such that

$$0 \longrightarrow \pi_*\mathcal{G}' \longrightarrow E_0 \longrightarrow E_1 \longrightarrow \mathrm{R}^1\pi_*\mathcal{G}' \longrightarrow 0, \tag{2.5.3}$$

where the E_i are locally free of finite rank. Then $\mathrm{d}_{\pi'}(\mathcal{G}') \simeq \det(E_0)\otimes\det(E_1)^\vee$.

Theorem 2.29 (Theorem 13.5.8 in [ACG11])**.** *If L and M are line bundles on $\pi : \mathcal{X} \to B$ then*

$$\langle L, M\rangle \simeq \mathrm{d}_\pi(L\otimes M)\otimes \mathrm{d}_\pi(L)^\vee \otimes \mathrm{d}_\pi(M)^\vee \otimes \mathrm{d}_\pi(\mathcal{O}_\mathcal{X}).$$

Lemma 2.30. *Let L be a line bundle of relative degree d on $\mathcal{X}$ and N be a line bundle on the base B. If B is reduced then*

$$\mathrm{d}_\pi(L\otimes\pi^*N) \simeq \mathrm{d}_\pi(L)\otimes N^{\otimes d-g+1}.$$

Proof. Note that $\mathrm{R}^i\pi_*(L\otimes\pi^*N) \simeq \mathrm{R}^i\pi_*(L)\otimes N$ for $i = 0, 1$ by the projection formula. If $\mathrm{R}^i\pi_*L$ is locally free of rank r_i for $i = 0, 1$ then $\det(\mathrm{R}^i\pi_*(L\otimes\pi^*N)) \simeq \det(\mathrm{R}^i\pi_*L)\otimes N^{\otimes r_i}$ by the projection formula. Thus:

$$\mathrm{d}_\pi(L\otimes\pi^*N) \simeq \mathrm{d}_\pi(L)\otimes N^{\otimes(r_0-r_1)}.$$

Since $\mathrm{R}^i\pi_*L$ are locally free, we may base change to a geometric fiber and apply Riemann–Roch to see that $r_0 - r_1 = d - g + 1$.

More generally, let us assume that on B we can construct an exact sequence as in 2.5.3 with E_i locally free of rank e_i. Tensoring this exact sequence with N we conclude that $\mathrm{d}_\pi(L \otimes \pi^* N) \simeq \det(E_0 \otimes N) \otimes \det(E_1 \otimes N)^\vee \simeq \mathrm{d}_\pi(L) \otimes N^{\otimes(e_0 - e_1)}$. Since B is reduced, after applying semi-continuity theorem we see that on a dense open set of B the pushforwards of L are locally free and that they respect base change. As before we may now conclude $e_0 - e_1 = r_0 - r_1 = d - g + 1$.

In complete generality, one has to base change on B to an étale cover and apply the previous construction. It is clear that the isomorphism thus constructed on the cover is natural and so it will descend to B. □

Corollary 2.31. *With the same hypotheses as in Lemma 2.30 we have*

$$\langle L, \pi^* N \rangle \simeq N^{\otimes d}.$$

Proof. Immediate upon substituting the result of Lemma 2.30 into Theorem 2.29. □

2.5.2 Intermediate steps

Remark 2.32. Since the components of $\overline{\mathcal{S}}_g^{--} \setminus \overline{\mathcal{U}}_g^{--}$ have codimension at least 2 the restriction map $\mathrm{CH}^1(\overline{\mathcal{S}}_g^{--}) \to \mathrm{CH}^1(\overline{\mathcal{U}}_g^{--})$ is an isomorphism. Therefore, the results will be stated in $\overline{\mathcal{S}}_g^{--}$ but the computations will solely be carried out in $\overline{\mathcal{U}}_g^{--}$ without mention. Furthermore, as $\overline{\mathcal{S}}_g^{--}$ and the universal quasi-stable curve over it are smooth the groups CH^1 and Pic are isomorphic. Over the coarse moduli spaces, $\mathrm{CH}^1_{\mathbb{Q}}$ and $\mathrm{Pic}_{\mathbb{Q}}$ are isomorphic.

Notation 2.33. With $\pi : \mathcal{C} \to \overline{\mathcal{S}}_g^{--}$ we set $\gamma := c_1(\omega_\pi)$ and $\kappa := \pi_*(\gamma^2)$ as is customary. Furthermore, we will let δ_i^{xy} stand for the isomorphism class of the line bundle $\mathcal{O}_{\overline{\mathcal{S}}_g^{--}}(\Delta_i^{xy})$. Finally, λ stands for the Hodge class $c_1(\pi_*\omega_\pi)$. It will be convenient to introduce the following two short hand notations:

$$\begin{aligned} \tilde{\delta}_i^{++} &:= \delta_i^{++} + \delta_i^{+=} + \delta_{g-i}^{-=} \\ \tilde{\delta}_i &:= \delta_i^{++} + \delta_i^{+=} + \delta_{g-i}^{-=} + \delta_i^{+-}. \end{aligned}$$

Lemma 2.34. *In* $\mathrm{Pic}_{\mathbb{Q}}(\overline{\mathcal{S}}_g^{--})$ *the κ class decomposes as:*

$$\kappa = 12\lambda - (\delta_0^{nn} + 2\delta_0^{nb} + 2\delta_0^{bn} + 2\delta_0^{bb} + 2\delta_0^{b=} + 2\sum_{i=1}^{g-1} \tilde{\delta}_i)$$

Proof. Pullback the formula $\kappa = 12\lambda - \delta$ from $\mathrm{Pic}_{\mathbb{Q}}(\overline{\mathcal{M}}_g)$, see [Mum83] and use the pullback formula for the boundary classes given in Equation 1.8.4. □

Remark 2.35. Applying Equation 2.5.2 gives us the following equality:

$$c_1\langle \mathcal{O}_{\mathcal{C}}(\mathcal{D}_1), \mathcal{O}_{\mathcal{C}}(\mathcal{D}_2)\rangle = \pi_*([\mathcal{D}_1]\cdot[\mathcal{D}_2]).$$

Since $\widetilde{\Omega}_g$ is defined to be the image $\pi_*(\mathcal{D}_1\cap\mathcal{D}_2)$ we conclude that the class above is effective and supported on $\widetilde{\Omega}_g$.

Recall that $\mathcal{O}_{\mathcal{C}}(\mathcal{D}_i)\simeq \tilde{\mathcal{L}}_i\otimes\pi^*(\pi_*\tilde{\mathcal{L}}_i)^\vee$. Plugging this into the Deligne pairing and using the bi-linearity of the pairing with respect to the tensor product we conclude that $\langle \mathcal{O}_{\mathcal{C}}(\mathcal{D}_1), \mathcal{O}_{\mathcal{C}}(\mathcal{D}_2)\rangle$ is isomorphic to the following:

$$\langle\tilde{\mathcal{L}}_1,\tilde{\mathcal{L}}_2\rangle\otimes\langle\pi^*\pi_*\tilde{\mathcal{L}}_1,\tilde{\mathcal{L}}_2\rangle^\vee\otimes\langle\tilde{\mathcal{L}}_1,\pi^*\pi_*\tilde{\mathcal{L}}_2\rangle^\vee\otimes\langle\pi^*\pi_*\tilde{\mathcal{L}}_1,\pi^*\pi_*\tilde{\mathcal{L}}_2\rangle. \tag{2.5.4}$$

Recalling that $\tilde{\mathcal{L}}_i$ is of relative degree $g-1$ we may apply Corollary 2.31 to the equation 2.5.4 to obtain:

$$\langle \mathcal{O}_{\mathcal{C}}(\mathcal{D}_1), \mathcal{O}_{\mathcal{C}}(\mathcal{D}_2)\rangle\simeq\langle\tilde{\mathcal{L}}_1,\tilde{\mathcal{L}}_2\rangle\otimes\left(\pi_*\tilde{\mathcal{L}}_1\otimes\pi_*\tilde{\mathcal{L}}_2\right)^{\otimes(1-g)}. \tag{2.5.5}$$

Let $f_i:\overline{\mathcal{S}}_g^{--}\to\overline{\mathcal{S}}_g^{-}$ be the functor which forgets the j-th spin structure, where $j\neq i$, and stabilizes the family. This morphism induces a map on the universal curves, which we will denote by F_i. By definition $\tilde{\mathcal{L}}_i = F_i^*\tilde{\mathcal{L}} = F_i^*(\mathcal{L}(-\sum_{j=1}^{g-1}C_j^+)) = \mathcal{L}_i(-\sum_{j=1}^{g-1}F_i^*(C_j^+))$.
Using bi-linearity we have:

$$\langle\tilde{\mathcal{L}}_1,\tilde{\mathcal{L}}_2\rangle = \langle\mathcal{L}_1,\mathcal{L}_2\rangle\otimes\langle\mathcal{L}_1,\sum_{i=1}^{g-1}F_2^*C_i^+\rangle^\vee\otimes\langle\sum_{i=1}^{g-1}F_1^*C_i^+,\mathcal{L}_2\rangle^\vee\otimes\langle\sum_{i=1}^{g-1}F_1^*C_i^+,F_2^*C_i^+\rangle. \tag{2.5.6}$$

Denote by C_i^{+x} the genus i component of the universal curve over Δ_i^{+x}. However, denote by $C_{g-i}^{-=}$ denote the *genus* i component over $\Delta_{g-i}^{-=}$. A more suitable, but cumbersome, name would have been $C_i^{++;=}$. We then have:

$$\begin{aligned} F_1^*C_i^+ &= C_i^{++}+C_i^{+-}+C_i^{+=}+C_{g-i}^{-=},\\ F_2^*C_i^+ &= C_i^{++}+C_i^{-+}+C_i^{+=}+C_{g-i}^{-=}. \end{aligned}$$

Lemma 2.36. *For each $i>0$ and in $\mathrm{Pic}(\overline{\mathcal{S}}_g^{--})_{\mathbb{Q}}$ we have:*

$$\langle\sum_{i=1}^{g-1}F_1^*C_i^+,F_2^*C_i^+\rangle\equiv-\sum_{i=1}^{g-1}(\delta_i^{++}+\delta_i^{+-}+\delta_i^{+=}+\delta_{g-i}^{-=}).$$

Proof. Notice that for $i\neq j, g-j$ the divisors C_i^{xy} and C_j^{st} will intersect over $\Delta_i^{xy}\cap\Delta_j^{st}$ which has codimension 2 in $\overline{\mathcal{S}}_g^{--}$. In particular $\pi_*([C_i^{xy}]\cdot[C_j^{st}])=0$.

Let E_i^{xy} be the exceptional divisor lying over Δ_i^{xy}. With $\pi:\mathcal{C}\to\overline{\mathcal{S}}_g^{--}$ the universal curve, we have $\pi^*[\Delta_i^{xy}]=[C_i^{xy}]+[E_i^{xy}]+[C_{g-i}^{\bar{x}\bar{y}}]$ we have:

$$\begin{aligned} \pi_*([C_i^{xy}]^2) &= \pi_*([C_i^{xy}]\cdot([\pi^*\Delta_i^{xy}]-[E_i^{xy}]-[C_{g-i}^{\bar{x}\bar{y}}]))\\ &= \pi_*([C_i^{xy}])\cdot[\Delta_i^{xy}]-\pi_*([C_i^{xy}]\cdot[E_i^{xy}]). \end{aligned}$$

For the last equality we used projection formula and the fact that C_i and C_{g-i} do not intersect. Note that the pushforward $\pi_*([C_i^{xy}])$ vanishes as C_i^{xy} is not finite over its image. On the other hand, it is easy to see that $\pi_*([C_i^{xy}]\cdot[E_i^{xy}])=\Delta_i^{xy}$. This gives:

$$\pi_*([C_i^{xy}]^2)=-[\Delta_i^{xy}].$$

Using $C_i^{xy}=C_{g-i}^{\bar{x}\bar{y}}$ we also have:

$$\pi_*([C_i^{xy}]\cdot[C_{g-i}^{\bar{x}\bar{y}}])=-[\Delta_i^{xy}].$$

Now we put all this together:

$$\pi_*([F_1^*C_i^+]\cdot[F_2^*C_j^+])=\begin{cases} -[\Delta_i^{++}]-[\Delta_i^{+=}]-[\Delta_{g-i}^{-=}] & : \quad i=j \\ -[\Delta_i^{+-}] & : \quad i=g-j \\ 0 & : \quad \text{otherwise} \end{cases}$$

Using bi-linearity and substituting the above equality gives the claimed result. □

Lemma 2.37. *For each $i>0$ we have the following identities in* $\mathrm{Pic}(\overline{\mathcal{S}}_g^{--})_{\mathbb{Q}}$*:*

$$\langle F_1^*C_i^+,\mathcal{L}_2\rangle\equiv(i-1)(\delta_i^{++}+\delta_i^{+=}+\delta_i^{+-}+\delta_{g-i}^{-=})$$
$$\langle\mathcal{L}_1,F_2^*C_i^+\rangle\equiv(i-1)(\delta_i^{++}+\delta_i^{+=}+\delta_i^{-+}+\delta_{g-i}^{-=})$$

Proof. Since $\mathcal{L}_1|_{C_i^{xy}}$ restricts on each fiber to a theta characteristic, the Chern class $c_1(\mathcal{L}_1|_{C_i^{xy}})$ has degree $i-1$ over Δ_i^{xy}. Therefore, $\pi_*(c_1(\mathcal{L}_1)\cdot[C_i^{xy}])=(i-1)[\Delta_i^{xy}]$. One argues similarly for $\mathcal{L}_2$. □

Lemma 2.38. *On* $\mathrm{Pic}(\overline{\mathcal{S}}_g^{--})_{\mathbb{Q}}$ *we have:*

$$\langle\mathcal{L}_1,\mathcal{L}_2\rangle=3\lambda-\frac{1}{4}\delta_0^{nn}-\frac{1}{2}\delta_0^{bn}-\frac{1}{2}\delta_0^{nb}-\delta_0^{bb}-\delta_0^{b=}-\sum_{i=1}^{g-1}\tilde{\delta}_i$$

Proof. Before we compute $c_1\langle\mathcal{L}_1,\mathcal{L}_2\rangle$ we first make the following observation. The squaring maps $\alpha_i:\mathcal{L}_i^{\otimes 2}\to\omega_\pi$ are naturally injective, since the ambient space is integral. Moreover, the cokernel of α_i is supported on the exceptional divisors on which $\mathcal{L}_i$ has non-zero degree, let us denote this divisorial locus by $E^{(i)}$. With this notation, we get $c_1(\mathcal{L}_j)=\frac{1}{2}(\gamma-[E^{(j)}])$. Notice that $c_1\langle\mathcal{L}_1,\mathcal{L}_2\rangle=\pi_*(c_1(\mathcal{L}_1)\cdot c_1(\mathcal{L}_2))=\frac{1}{4}\pi_*(\gamma^2-\gamma\cdot[E^{(1)}]-\gamma\cdot[E^{(2)}]+[E^{(1)}]\cdot[E^{(2)}])$.

By definition the ideal sheaf of $E^{(i)}$ is the image of $\mathcal{L}_i^{\otimes 2}\otimes\omega_\pi^\vee\hookrightarrow\mathcal{O}_{\mathcal{C}}$ and therefore $\mathcal{O}_{\mathcal{C}}(E^{(i)})\simeq\omega_\pi\otimes\mathcal{L}^{\otimes -2}$.

In terms of familiar symbols we can write $E^{(j)}$ as follows:

$$E^{(1)}=E_0^{bb}+E_0^{b=}+E_0^{bn}+\sum_{i=1}^{g-1}(E_i^{++}+E_i^{+=}+E_i^{+-}+E_{g-i}^{-=})$$
$$E^{(2)}=E_0^{bb}+E_0^{b=}+E_0^{nb}+\sum_{i=1}^{g-1}(E_i^{++}+E_i^{+=}+E_i^{+-}+E_{g-i}^{-=})$$

Clearly E_i^{xy} and $E_j^{x'y'}$ will not intersect unless $i=j$ and $xy=x'y'$. In particular, for $i \neq 0$ we have $[E_i^{xy}]^2 = c_1(\mathcal{O}_{\mathcal{C}}(E^{(1)})|_{E_i^{xy}})$, where the Chern class is computed on E_i^{xy} and pushed forward onto $\mathcal{C}$. But of course $\mathcal{O}_{\mathcal{C}}(E^{(1)})|_{E_i^{xy}} \simeq (\omega_\pi \otimes \mathcal{L}^{\otimes -2})|_{E_i^{xy}} \simeq \mathcal{O}_{E_i^{xy}}(-2)$, where we recall that E_i^{xy} is a $\mathbb{P}^1$-bundle over Δ_i^{xy}. In particular, $\pi_*([E_i^{xy}]^2) \equiv -2\delta_i^{xy}$.

Clearly ω_π restricts to a line bundle of relative degree 0 on each E_i^{xy}, therefore $\pi_*(\gamma \cdot [E^{(j)}]) = 0$. Putting all this together we have:

$$c_1\langle \mathcal{L}_1, \mathcal{L}_2\rangle = \frac{1}{4}(\kappa - 2\delta_0^{bb} - 2\delta_0^{b=} - 2\sum_{i=1}^{g-1}\tilde{\delta}_i)$$

Substituting Mumford's formula for κ on $\overline{\mathcal{M}}_g$ and pulling it back to $\overline{\mathcal{S}}_g^{--}$ we obtain the desired result. $\square$

Corollary 2.39. *In* $\mathrm{Pic}(\overline{\mathcal{S}}_g^{--})_{\mathbb{Q}}$ *we have:*

$$c_1\langle \tilde{\mathcal{L}}_1, \tilde{\mathcal{L}}_2\rangle \equiv 3\lambda - \frac{1}{4}\delta_0^{nn} - \frac{1}{2}\delta_0^{bn} - \frac{1}{2}\delta_0^{nb} - \delta_0^{bb} - \delta_0^{b=} - \sum_{i=1}^{g-1}(2i-1)\tilde{\delta}_i^{++} - (g-1)\sum_{i=1}^{g-1}\delta_i^{+-}.$$

Proof. Formula 2.5.6 expresses $\langle \tilde{\mathcal{L}}_1, \tilde{\mathcal{L}}_2\rangle$ in terms of 4 tensor products. Reading from right to left, the Chern class of each of these 4 terms are computed in Lemmas 2.36, 2.37 and 2.38. We simply plugin each of these results. The only subtlety of this substitution is the observation that $(i-1)\delta_i^{-+} = (i-1)\delta_{g-i}^{+-}$ which accounts for the final term. $\square$

Lemma 2.40. *On* $\mathrm{Pic}(\overline{\mathcal{S}}_g^{-})_{\mathbb{Q}}$ *we have the following relation:*

$$c_1(\pi_*\tilde{\mathcal{L}}) \equiv -\frac{1}{4}\lambda + \frac{1}{16}\delta_0^n.$$

Proof. Cornalba proves that $\mathrm{d}_\pi(\mathcal{L}) = -\frac{1}{2}\lambda + \frac{1}{8}\delta_0^n$ in Theorem 3.6 [Cor91]. We therefore need only show that $\mathrm{d}_\pi(\mathcal{L}) \simeq (\pi_*\mathcal{L})^{\otimes 2}$ and $\pi_*\tilde{\mathcal{L}} \simeq \pi_*\mathcal{L}$ to complete the proof.

On $\overline{\mathcal{U}}_g^-$ the sheaves $\pi_*\mathcal{L}$ and $\mathrm{R}^1\pi_*\mathcal{L}$ are both locally free of rank 1, thus $\mathrm{d}_\pi(\mathcal{L}) \simeq \pi_*(\mathcal{L}) \otimes (\mathrm{R}^1\pi_*\mathcal{L})^\vee$. By relative Serre duality we have $(\mathrm{R}^1\pi_*\mathcal{L})^\vee \simeq \pi_*(\omega_\pi \otimes \mathcal{L}^\vee)$. Recall that $\omega_\pi \otimes \mathcal{L}^\vee \simeq \mathcal{L}(E)$ where E is the divisor consisting of all exceptional components of singular fibers. We claim $\pi_*(\mathcal{L}(E)) \simeq \pi_*(\mathcal{L})$.

Consider the natural exact sequence

$$0 \longrightarrow \mathcal{L} \longrightarrow \mathcal{L}(E) \longrightarrow \mathcal{L}(E)|_E \longrightarrow 0$$

and observe that $\mathcal{L}|_E$ has relative degree 1 on the $\mathbb{P}^1$-bundle E whereas, as we computed before, $\mathcal{O}(E)|_E$ has relative degree -2. Therefore $\mathcal{L}(E)|_E$ has relative degree -1, and its pushforward is 0. Pushing forward the exact sequence thus proves $\pi_*(\mathcal{L}(E)) \simeq \pi_*(\mathcal{L})$.

It remains to prove $\pi_*\tilde{\mathcal{L}} \simeq \pi_*\mathcal{L}$. Let C^+ stand for the sum $\sum_{i=1}^{g-1} C_i^+$ consisting of all components of singular fibers where $\mathcal{L}$ is even. By definition $\tilde{\mathcal{L}} = \mathcal{L}(-C^+)$ which gives us the following exact sequence:

$$0 \longrightarrow \tilde{\mathcal{L}} \longrightarrow \mathcal{L} \longrightarrow \mathcal{L}|_{C^+} \longrightarrow 0$$

By definition of $\overline{\mathcal{U}}_g^-$ and of C^+, it is clear that $\mathcal{L}|_{C^+}$ restricts on each fiber to an even theta characteristic without any global sections. Using the fact that Δ_i^{xy} are reduced, we conclude by cohomology and base change that $\pi_*(\mathcal{L}|_{C^+}) = 0$. The same argument holds for the higher pushforward, giving $\mathrm{R}^1\pi_*(\mathcal{L}|_{C^+}) = 0$. Therefore $\pi_*\tilde{\mathcal{L}} \simeq \pi_*\mathcal{L}$ and $\mathrm{R}^1\pi_*\tilde{\mathcal{L}} \simeq \mathrm{R}^1\pi_*\mathcal{L}$. □

Corollary 2.41. *In* $\mathrm{Pic}(\overline{\mathcal{S}}_g^{--})_{\mathbb{Q}}$ *we have:*

$$\begin{aligned} c_1(\pi_*\tilde{\mathcal{L}}_1) \equiv & -\frac{1}{4}\lambda + \frac{1}{16}\delta_0^{nn} + \frac{1}{8}\delta_0^{nb}, \\ c_1(\pi_*\tilde{\mathcal{L}}_2) \equiv & -\frac{1}{4}\lambda + \frac{1}{16}\delta_0^{nn} + \frac{1}{8}\delta_0^{bn}. \end{aligned}$$

Proof. Since $\tilde{\mathcal{L}}_i$ is the pullback of $\tilde{\mathcal{L}}$ via the forgetful functor F_i and since on $\overline{\mathcal{U}}_g^-$ the line bundle $\pi_*\tilde{\mathcal{L}}$ satisfies cohomology and base change, we conclude $f_i^*([\pi_*\tilde{\mathcal{L}}]) \equiv [\pi_*\tilde{\mathcal{L}}_i]$. Now use Lemma 2.40 and the pullback formulae on boundary divisors to get the stated formulae. □

2.5.3 Conclusion

Notation 2.42. Let $\mathcal{Z}_g \in \mathrm{Pic}_{\mathbb{Q}}(\overline{\mathcal{S}}_g^{--})$ stand for the divisor class $\pi_*([\mathcal{D}_1]\cdot[\mathcal{D}_2])$. Recall, $\tilde{\delta}_i^{++} = \delta_i^{++} + \delta_i^{+=} + \delta_{g-i}^{-=}$.

Theorem 2.43. *In* $\mathrm{Pic}(\overline{\mathcal{S}}_g^{--})_{\mathbb{Q}}$ *the class* $\mathcal{Z}_g$ *is equivalent to*

$$\frac{g+5}{2}\lambda - \frac{g+1}{8}\delta_0^{nn} - \frac{g+3}{8}\left(\delta_0^{nb} + \delta_0^{bn}\right) - \delta_0^{bb} - \delta_0^{b=} - \sum_{i=1}^{g-1}(2i-1)\tilde{\delta}_i^{++} - (g-1)\sum_{i=1}^{g-1}\delta_i^{+-}.$$

Proof. Equation 2.5.5 allows us to express $c_1\langle\mathcal{O}_{\mathcal{C}}(\mathcal{D}_1), \mathcal{O}_{\mathcal{C}}(\mathcal{D}_2)\rangle = \pi_*([\mathcal{D}_1]\cdot[\mathcal{D}_2])$ in terms of the Chern classes $c_1\langle\tilde{\mathcal{L}}_1, \tilde{\mathcal{L}}_2\rangle$, $c_1(\pi_*\tilde{\mathcal{L}}_1)$ and $c_1(\pi_*\tilde{\mathcal{L}}_2)$. The first of these is computed in Corollary 2.39 while the last two are computed in Corollary 2.41. We put all these together to obtain the result. □

Lemma 2.44. *The two closures of* Ω_g *are related in the following way:*

$$[\widetilde{\Omega}_g] = [\overline{\Omega}_g] + \delta_0^{b=} + \sum_{i=2}^{g-1}\left(\delta_i^{-=} + \delta_{g-i}^{+=}\right)$$

Proof. This statement is expected but in fact not easy to prove. One has to rule out that $\widetilde{\Omega}_g$ does not contain other boundary components. This is done in Theorem 3.1 from which this result follows. □

We computed the effective divisor class $\mathcal{Z}_g$ in order to compute $[\overline{\Omega}_g]$. However, $\mathcal{Z}_g$ clearly is a larger class as it contains in excess the boundary components contained in $\widetilde{\Omega}_g$. There is another problem however. We don't know if there is generically one contact point on every double spin curve in $\overline{\Omega}_g$. If there were more, then the pushforward $\pi_*(\mathcal{D}_1 \cap \mathcal{D}_2)$ would over count some components of $\overline{\Omega}_g$.

However, we discussed in the Preface that we expect $\overline{\Omega}_g$ to be irreducible and that if it is irreducible then the general curve in $\overline{\Omega}_g$ will have a single contact point. With this we may conclude the following:

Corollary 2.45. *If $\overline{\Omega}_g$ is irreducible then in* $\mathrm{Pic}(\overline{\mathcal{S}}_g^{--}) \otimes \mathbb{Q}$ *we have the following relation:*

$$[\overline{\Omega}_g] = \mathcal{Z}_g - (g-2)\delta_0^{b=} - \sum_{i=1}^{g-1}(i-1)\delta_i^{-=} - \sum_{i=1}^{g-1} i\delta_i^{+=}$$

Proof. The assumption on irreducibility is to ensure that $\mathcal{Z}_g|_{\mathcal{S}_g^-} \equiv [\Omega_g]$, as we discussed above. Of course, it must be shown that $\mathcal{D}_1$ and $\mathcal{D}_2$ intersect generically transversally over Ω_g. However, having assumed Ω_g is irreducible, it is sufficient to check this over one point. This is easiest over a boundary point in $\overline{\Omega}_g$. Using the classification in Chapter 3 we can restrict within Δ_1^{++} to the locus which is essentially Ω_{g-1}. There we can apply induction to see that the normal directions of $\mathcal{D}_i$ do not coincide.

To prove the claim, it remains to describe the excess of $\mathcal{Z}_g$ when compared to $\overline{\Omega}_g$. Note that once again Theorem 3.1 forms the basis of argument, since it determines the divisor classes for which we expect excess. Let us explain the $\delta_0^{b=}$ coefficient in detail, the others being similar.

The locus $\Delta_0^{b=}$ parametrizes curves (X, L'_η, L''_η) as described in Example 1.33. Let $C \hookrightarrow X$ be the stable component of X. Then the contact divisors of L'_η and L''_η agree on C and they are both η, which is of degree $g-2$. The $(g-1)$-th point for both theta characteristics lie on the exceptional component and they do not coincide in general. Therefore, $\mathcal{D}_1 \cap \mathcal{D}_2$ has one (or possibly more) component(s) over $\Delta_0^{b=}$ of total degree $g-2$. We also showed that $\mathcal{D}_1$ and $\mathcal{D}_2$ intersect generically transversally at every point over $\Delta_0^{b=}$, see Corollary 3.26. □

2.6 Towards Kodaira dimension

Remark 2.46. For the entirety of this section we will be working with the coarse moduli schemes associated to the stacks we have been dealing with. Moreover, let $\pi : \overline{S}_g^{--} \to \overline{M}_g$ denote the usual map.

Farkas and Verra in [FV14] complete the Kodaira classification of $\overline{S}_g^-$. In particular, they show that $\overline{S}_g^-$ is of general type for $g \geq 12$ and is unirational for $g < 12$. Since $\overline{S}_g^{--} \to \overline{S}_g^-$ is a finite cover of large degree, one would expect $\overline{S}_g^{--}$ to be of general type starting from smaller genus. We will prove the following:

Theorem 2.47. *The canonical class on $\overline{S}_g^{--}$ is big if $g \geq 10$.*

Showing that the canonical class is big is half the battle towards showing that the space is of general type. For example, with $\overline{M}_g$, $\overline{S}_g^-$ and $\overline{S}_g^+$ the singularities are analyzed and it is shown that the pluricanonical classes on these singular spaces lift to any desingularization, see [HM82] and [Lud10]. If this property holds for $\overline{S}_g^{--}$ then we may also conclude $\overline{S}_g^{--}$ is of general type whenever the canonical class is big.

Corollary 2.48. *If pluricanonical forms on $\overline{S}_g^{--}$ lift to any desingularization then $\overline{S}_g^{--}$ is of general type when $g \geq 10$.*

Remark 2.49. In fact, $\overline{S}_g^{--}$ is of general type, unconditionally, starting from $g \geq 12$. Indeed, $\overline{S}_g^{-}$ is of general type for $g \geq 12$ and $\overline{S}_g^{--} \to \overline{S}_g^{-}$ is finite so we may apply Corollary 9 of [Kaw81] to a morphism between two desingularizations.

To prove the theorem let us recall the standard strategy. The Hodge class $\lambda \in \mathrm{Pic}_{\mathbb{Q}}(\overline{S}_g^{--})$ is big, because it is the pullback of the Hodge class in $\mathrm{Pic}_{\mathbb{Q}}(\overline{M}_g)$ which is big (see [HM82]). Let $K_g \in \mathrm{Pic}_{\mathbb{Q}}(\overline{S}_g^{--})$ be the canonical class and suppose that we can write:

$$K_g \equiv a\lambda + D, \tag{2.6.7}$$

where $a > 0$ and D is an effective divisor. Then, K_g is big. The test for the existence of such a decomposition can further be simplified using the usual notion of a *slope*, which we now describe for $\overline{S}_g^{--}$.

Notation 2.50. Let $V = \mathbb{Q}\langle \lambda, \delta_0^{nn}, \delta_0^{nb}, \dots \rangle$ be the $\mathbb{Q}$-vector subspace of $\mathrm{Pic}_{\mathbb{Q}}(\overline{S}_g^{--})$ generated by the Hodge class and the boundary classes. For a divisor class $D = a_\lambda \lambda - \sum_{i,x,y} a_i^{xy} \delta_i^{xy} \in V$ we define:

$$s_i^{xy}(D) = \begin{cases} \frac{a_\lambda}{a_i^{xy}} & : \ a_i^{xy} \neq 0 \\ \infty & : \ a_i^{xy} = 0 \end{cases}$$

Definition 2.51. Given $D, D' \in V$, if for all i, x, y we have $s_i^{xy}(D) < s_i^{xy}(D')$ then we will say *D has smaller slope than D'*.

Remark 2.52. The symbols s_i^{xy} are well defined only if the generators of V are linearly independent. However, this is true and can be proven using the line of reasoning given in [Cor89], which relies on test curves. The argument is elementary and we do not repeat it here.

It is straightforward to check that if there exists an effective divisor $Q_g \in \mathrm{Pic}_{\mathbb{Q}}(\overline{S}_g^{--})$ such that Q_g has smaller slope than K_g then we can find $a > 0$ and effective D such that the expression 2.6.7 holds, and thus K_g is big. Now we will construct such Q_g for $g = 10, 11$.

When $g = 10$ consider the divisor $\overline{\mathcal{K}} \subset \overline{M}_{10}$ generically consisting of curves lying on K3 surfaces. Farkas and Popa [FP05] compute the class of this divisor in terms of the standard generators giving:

$$\overline{\mathcal{K}} \equiv 7\lambda - \delta_0 - 5\delta_1 - 9\delta_2 - 12\delta_3 - 14\delta_4 - B_5\delta_5, \qquad B_5 \geq 6.$$

Lemma 2.53. *The class $Q_{10} := 2\pi^*\overline{\mathcal{K}} + \mathcal{Z}_{10}$ has smaller slope than K_{10} in* $\mathrm{Pic}_{\mathbb{Q}}(\overline{\mathcal{S}}_{10}^{--})$.

Proof. Using our Theorem 2.43 and the expression of the pullbacks from Equation 1.8.4, we can write the class Q_{10} explicitly in terms of the standard generators. Using Corollary 1.61 to compute the slope of the canonical class K_{10} we see that Q_{10} has smaller slope than K_{10}. $\square$

For $g = 11$ we will use the Brill–Noether divisor $\overline{M}^r_{g,d} \subset \overline{M}_g$ just like [FV14]. Theorem 1 of [EH87] states that there is a positive rational constant $c_{g,r,d}$ such that the following equality holds in $\mathrm{Pic}_{\mathbb{Q}}(\overline{M}_g)$:

$$[\overline{M}^r_{g,d}] \equiv c_{g,r,d} \left((g+3)\lambda - \frac{g+1}{6}\delta_0 - \sum_{i=1}^{\lfloor \frac{g}{2} \rfloor} i(g-i)\delta_i \right). \tag{2.6.8}$$

Lemma 2.54. *The class* $Q_{11} := \frac{3}{c_{11,1,6}}\pi^*[\overline{M}^1_{11,6}] + 2\mathcal{Z}_{11}$ *has smaller slope than* K_{11} *in* $\mathrm{Pic}_{\mathbb{Q}}(\overline{\mathcal{S}}^{--}_{11})$.

Proof. Once again, use the pullback formulae, the formula for $\mathcal{Z}_{11}$ and the formula for the canonical class to write the classes explicitly whereupon the result is immediate. $\square$

Proof of Theorem 2.47. Farkas and Verra prove that $\overline{S}^-_g$ is of general type for $g \geq 12$ by showing that the canonical class in this range is big. Since $\overline{S}^{--}_g \to \overline{S}^-_g$ is finite, we can use the Riemann–Hurwitz exact sequence to show that the canonical class on $\overline{S}^{--}_g$ is also big when $g \geq 12$. For $g = 10$ and $g = 11$ the $\mathbb{Q}$-effective classes Q_{10} and Q_{11} have slope less than K_{10} and K_{11}, respectively, hence the theorem follows. $\square$

Remark 2.55. In fact, for $g \geq 11$ and $g+1$ composite we can define $Q_g = \frac{3(4g-31)}{c_{g,r,d}}[\overline{M}^r_{g,d}] + 2(g+1)\mathcal{Z}_g$ and then Q_g will have slope smaller than K_g in this range. For $g \geq 11$ and $g+1$ prime, one can instead use the Petri divisors computed in Theorem 2 [EH87]. This gives a proof of Theorem 2.47 that does not rely on [FV14].

Chapter 3

Degenerate common contact curves

The purpose of this chapter is to study Ω_g in detail, using its Zariski closure $\overline{\Omega}_g$. However, it's simpler to work with the modular closure $\widetilde{\Omega}_g$ of Ω_g, so our first goal is to find the locus where these two closures agree, see Theorem 3.1.

The second goal is to describe the locus $\overline{\Omega}_g \setminus \Omega_g$, i.e., the boundary of $\overline{\Omega}_g$. For our purposes we are interested only in the simplest nodal curves that appear in the boundary of $\overline{\Omega}_g$ and so we give a classification of these in Section 3.1.1.

For most of the boundary components Δ_i^{xy}, it is easy to guess what $\overline{\Omega}_g \cap \Delta_i^{xy}$ should be. These are exactly the boundary components where $\overline{\Omega}_g$ and $\widetilde{\Omega}_g$ agree. On the other hand, there are boundary components, namely $\Delta_i^{x=}$, wholly contained in $\widetilde{\Omega}_g$ but not in $\overline{\Omega}_g$. Finding the intersection $\overline{\Omega}_g \cap \Delta_i^{x=}$ is therefore much harder: we can no longer rely on the modular interpretation of $\widetilde{\Omega}_g$ and limit linear series arguments don't appear to work here. The tools we construct in Section 3.3.1 allow us to resolve the issue. This approach seems to be new and may find wider application.

Our third goal has been to see if $\overline{\Omega}_g$ is of degree 2 over its image in $\overline{\mathcal{M}}_g$. If true, this implies that the general curve admitting a pair of theta hyperplanes with a common contact point, admits only one such pair. Furthermore, the pushforward of the class $[\overline{\Omega}_g]$ in $\overline{\mathcal{M}}_g$ would give a new effective class that can be immediately be computed from Theorem 2.45.

We have not succeeded in achieving this last goal in its entirety, tough much progress is made: most of this chapter and a part of the appendix is dedicated entirely to this problem. We state the current state of the problem in the next section.

3.1 Summary of results

Theorem 3.1. *The two closures of Ω_g are related as follows:*

$$\widetilde{\Omega}_g = \overline{\Omega}_g \cup \Delta_0^{b=} \bigcup_{i=2}^{g-1} \left(\Delta_{g-i}^{+=} \cup \Delta_i^{-=} \right)$$

Proof. Both closures are pure of codimension 1 and both agree away from the boundary. Therefore, $\widetilde{\Omega}_g$ must be the union of $\overline{\Omega}_g$ with some boundary components. It remains to find which boundary components these are. By construction, none of the boundary components are contained in $\overline{\Omega}_g$. Whereas $\Delta_i^{xy} \subset \widetilde{\Omega}_g$ iff for the generic double spin curve in Δ_i^{xy} its two contact divisors intersect.

It is immediate that $\Delta_0^{b=}$, $\Delta_i^{+=}$ and $\Delta_i^{-=}$ all appear in $\widetilde{\Omega}_g$ with the exception of $\Delta_1^{-=}$ since the odd theta characteristic on an elliptic curve has no contact point to share. Recalling from Remark 1.47 that $\Delta_{g-1}^{+=} = \emptyset$, we can express the union of these boundary components as we have done so in the statement of the theorem.

We show that none of the other boundary components appear in $\widetilde{\Omega}_g$ by proving that the generic element in Δ_i^{xy} has disjoint contact divisors. This requires a different argument for each boundary component, which we provide one by one throughout the chapter. $\square$

Let $\overline{X} \in \overline{\mathcal{M}}_g$ be a stable curve and let $F_{\overline{X}} = \pi^{-1}(\overline{X})$ be the fiber of $\pi : \overline{\mathcal{S}}_g^{--} \to \overline{\mathcal{M}}_g$ over $\overline{X}$. Assuming $\overline{\Omega}_g$ is irreducible, to show $\overline{\Omega}_g$ is of degree 2 over its image in $\overline{\mathcal{M}}_g$ we would need only find one $\overline{X}$ such that $\#(\overline{\Omega}_g \cap F_{\overline{X}}) = 2$. This follows from the semi-continuity theorem because $\overline{\Omega}_g$ is symmetric with respect to the ordering of the two spin structures, so that the degree is at least 2.

To do this, we pick $\overline{X} \in \Delta_0$ of the type which we call a theta nodal curve (see Definition 3.10). The locus in $\overline{\mathcal{S}}_g^{--}$ over Δ_0 decomposes into five pieces: $\Delta_0^{bb}, \Delta_0^{b=}, \Delta_0^{bn}, \Delta_0^{nb}, \Delta_0^{nn}$. Therefore, we can break the problem of computing $\#(\overline{\Omega}_g \cap F_{\overline{X}})$ in to five by computing each $f_{\overline{X}}^{xy} := \#(\overline{\Omega}_g \cap \Delta_0^{xy} \cap F_{\overline{X}})$ separately. We prove the following:

Theorem 3.2. *If $\overline{X} \in \Delta_0$ is a general theta nodal curve then $f_{\overline{X}}^{bb} = 2$, $f_{\overline{X}}^{b=} = 0$, $f_{\overline{X}}^{nb} = 0$, $f_{\overline{X}}^{bn} = 0$.*

Proof. Proposition 3.12 implies that for a theta nodal curve $\overline{X}$ we have $f_{\overline{X}}^{bb} \neq 0$. On the other hand, Proposition 3.16 implies that $f_{\overline{X}}^{bb} = 2$.

To see that $f_{\overline{X}}^{bn} = 0$ we need to show that given a theta nodal curve $\overline{X}$ and *any* element $(\overline{X}, \mathcal{E}_1, \mathcal{E}_2) \in \Delta_0^{bn}$ we must have $(\overline{X}, \mathcal{E}_1, \mathcal{E}_2) \notin \overline{\Omega}_g$. Since $\overline{\Omega}_g$ and $\widetilde{\Omega}_g$ agree over Δ_0^{bn} (Remark 3.33) we need to show that the limit contact points of $\mathcal{E}_1$ and $\mathcal{E}_2$ are disjoint. This is the content of Proposition 3.37. Since Δ_0^{bn} and Δ_0^{nb} are symmetric, this also implies $f_{\overline{X}}^{nb} = 0$.

In order to prove $f_{\overline{X}}^{b=} = 0$ the reasoning would be similar to proving $f_{\overline{X}}^{nb} = 0$. The exception is that $\Delta_0^{b=} \subset \widetilde{\Omega}_g$ but $\Delta_0^{b=} \not\subset \overline{\Omega}_g$. So we first devise an explicit criterion for checking when an element $\xi \in \Delta_0^{b=}$ belongs to $\overline{\Omega}_g$, this is Proposition 3.21. Then we apply this criterion to show that if the underlying curve of ξ is general theta nodal then $\xi \notin \overline{\Omega}_g$, this follows from Proposition 3.29. $\square$

It remains to compute $f_{\overline{X}}^{nn}$. Although we could not compute this number, we reduce it to something easy to check, which we now describe. First note that, if C is a hyperelliptic curve and $w_1, w_2 \in C$ are two distinct Weierstrass points then Lemma 3.31 implies that $\forall \tau \in \sqrt{\omega_C(w_1 + w_2)}$ we have $h^0(\tau) = 1$ so that the following statement makes sense.

Conjecture 3.3. *For each $g \geq 2$ there is a hyperelliptic curve C of genus g with two Weierstrass points w_1, w_2 such that for any distinct pair $\tau_1, \tau_2 \in \sqrt{\omega_C(w_1 + w_2)}$ we have $\tau_1 \cap \tau_2 = \emptyset$.*

Remark 3.4. We will provide some evidence for this conjecture. First, notice that $\omega_C(w_1 + w_2)$ gives a nodal embedding of C into $\mathbb{P}^g$ with w_1 and w_2 mapping to the node. Then each $\tau \in \sqrt{\omega_C(w_1 + w_2)}$ corresponds to a contact hyperplane of this curve (which does not pass through the node). When $g = 2$ these contact hyperplanes are bitangents and therefore they can not share common points of contact. For higher genera, one would proceed by induction by degenerating the curve to a nodal curve of type $i > 0$. Unfortunately, the induction hypothesis is not sufficient to cover all possible degenerations of the pair (τ_1, τ_2). If one could show that each component of the moduli space of tuples $(C, w_1, w_2, \tau_1, \tau_2)$ contained the type of degeneration where we could apply induction, then the conjecture would be proven.

Corollary 3.5. *If Conjecture 3.3 is true, then $f^{nn}_{\overline{X}} = 0$. If, moreover, $\overline{\Omega}_g$ is irreducible then $\overline{\Omega}_g \to \overline{\mathcal{M}}_g$ is degree 2 over its image.*

3.1.1 Classification of limit contact curves

In this section, we will list the *general* boundary curves in $\overline{\Omega}_g$. These results follow from the rest of this chapter and we cite the relevant parts.

Global assumptions. Throughout this subsection we assume that $\overline{X} \in \Delta \subset \overline{\mathcal{M}}_g$ has a single node x and all theta characteristics to appear (limit or otherwise) are rigid.

Notation 3.6. Let $\nu : C \to \overline{X}$ be the normalization map and $\nu^{-1}(x) = \{p, q\}$. If $X \in \Delta_i$ with $i > 0$ then $C = C_1 \sqcup C_2$ with $g(C_1) = i$ and $p \in C_1$. We use the notation introduced in Section 1.6 for the double spin structures on $\overline{X}$; each of the cases considered below are covered by the examples there.

Over Δ_0^{bb}: Let $\xi' = (\overline{X}, (\eta_1, \eta_2)')$ and $\xi'' = (\overline{X}, (\eta_1, \eta_2)'')$ be in Δ_0^{bb} (see Example 1.26 for the notation). If $(C, \eta_1, \eta_2) \in \Omega_{g-1}$, that is if $\eta_1 \cap \eta_2 \neq \emptyset$, and $p, q \not\leq \eta_i$ then both ξ' and ξ'' belong to $\overline{\Omega}_g \cap \Delta_0^{bb}$.

Assume now that $\eta_1 \cap \eta_2 = \emptyset$. If $(p, q) \in C^2$ is a theta marking associated to (η_1, η_2) (see Definition 3.8 and Lemma 3.14) then precisely one of ξ' and ξ'' lies in $\overline{\Omega}_g \cap \Delta_0^{bb}$ (Proposition 3.12). Otherwise, neither ξ' nor ξ'' lies in $\overline{\Omega}_g \cap \Delta_0^{bb}$.

Over $\Delta_0^{b=}$: Let $\xi = (\overline{X}, (\eta, \eta)')$ be as in Example 1.27. We assume $(C, \eta, p, q) \in \mathcal{S}^-_{g-1,2}$ to be general in the sense that $p, q \not\leq \eta$ and $\mathfrak{d} \in |\eta|$ is reduced. Then $\xi \in \overline{\Omega}_g \cap \Delta_0^{b=}$ iff $\eta(p + q)$ has base points (Proposition 3.21).

Over Δ_0^{bn},Δ_0^{nb}, Δ_0^{nn}: The non-singular theta characteristics will not have contact points along the exceptional divisor of $X \to \overline{X}$. Therefore, the problem reduces to checking the supports of the relevant roots on $(C, p, q) \in \mathcal{M}_{g-1,2}$. We analyze the situation further in Sections 3.4 and 3.5.

Over Δ_i^{++}: We denote the isomorphism class of $\xi = (\overline{X}, \mathcal{E}_1, \mathcal{E}_2) \in \Delta_i^{++}$ by $(\overline{X}, \Xi)$ such that $\Xi = \left(\begin{smallmatrix} \eta_{11} & \eta_{12} \\ \eta_{21} & \eta_{22} \end{smallmatrix}\right)$, see Example 1.30. In his case $\overline{X} = C_1 \cup_{p \sim q} C_2$ and $\eta_{11}, \eta_{12} \in S_{C_1}^+$ are distinct. In Section 3.6.1 we prove that:

- If $(C_2, \eta_{21}, \eta_{22}) \in \Omega_{g-i}$ and $q \not\leq \eta_{2j}$ then $\xi \in \overline{\Omega}_g \cap \Delta_i^{++}$.
- If $(C_2, \eta_{21}, \eta_{22}) \notin \Omega_{g-i}$ then $\xi \in \overline{\Omega}_g \cap \Delta_i^{++}$ iff $(\eta_{11} + p) \cap (\eta_{12} + p) \neq \emptyset$. In other words, $p \in C$ must be a Scorza point corresponding to (η_{11}, η_{12}) (see Definition 3.43).

Over Δ_i^{+-}: Let $\xi \in \Delta_i^{+-}$ be denoted by $(\overline{X}, \Xi)$ as above. This time $\eta_{11} \in S_C^+$ but $\eta_{12} \in S_C^-$. We prove in Section 3.6.2 that $\xi \in \overline{\Omega}_g \cap \Delta_i^{+-}$ iff one of the following hold: either $(\eta_{11} + p) \cap \eta_{12} \neq \emptyset$ or $\eta_{21} \cap (\eta_{22} + q) \neq \emptyset$. In other words, either p must be a switch point corresponding to (η_{11}, η_{12}) or q must be a switch point corresponding to (η_{22}, η_{21}) (see Definition 3.41).

Over $\Delta_i^{+=}$ and $\Delta_i^{-=}$: The methods developed for $\Delta_0^{b=}$ also work in this context. But the computations are harder still and will not be given here.

3.2 Theta nodal curves

Let C be a smooth curve with two distinct odd theta characteristics η_1 and η_2 such that $h^0(\eta_i) = 1$. For each $i = 1, 2$ let $H_i \subset \mathbb{P}(\mathrm{H}^0(C, \omega_C))$ be the hyperplane corresponding to η_i and let $\Gamma = H_1 \cap H_2$. There is a pencil of hyperplanes containing Γ.

Definition 3.7. If H is a hyperplane containing $\Gamma = H_1 \cap H_2 \subset \mathbb{P}(\mathrm{H}^0(C, \omega_C))$ then the canonical divisor $H \cdot C$ will be called *an associated divisor of* (H_1, H_2) (*or of* (η_1, η_2)). The space of associated divisors of (η_1, η_2) will be denoted by $[\eta_1, \eta_2]$.

Definition 3.8. Let D be an associated divisor of (η_1, η_2). If $(p, q) \in C^2$ is a pair of points such that $p + q \leq D$ then the tuple (p, q) will be called a *theta marking on C associated to* (η_1, η_2). If $p + q \leq \eta_1$, $p + q \leq \eta_2$ or $p = q$ then (p, q) will be called a *degenerate* theta marking and otherwise a *non-degenerate* theta marking.

Remark 3.9. We prove in Proposition 3.16 that on a general curve C, all but finitely many of the theta markings (p, q) are associated to a *unique* pair of theta characteristics.

Definition 3.10. If a stable nodal curve $\overline{X} \in \Delta_0 \subset \overline{\mathcal{M}}_g$ is obtained by clutching a smooth curve C at a theta marking (p, q) (associated to (η_1, η_2)) then we will call $\overline{X}$ a *theta nodal curve* (*associated to* (η_1, η_2)).

Proposition 3.12 below justifies why most of this chapter is devoted to a study of theta nodal curves.

Definition 3.11. A curve $C \in \overline{\mathcal{M}}_g$ will be called *theta general* if every theta characteristic on C has either 0 or 1 dimensional global sections. If $\overline{X} \in \Delta_0$ is a nodal curve whose normalization is theta general then we will say X is theta general.

Let $C \in \mathcal{M}_{g-1}$ be theta general curve with $p, q \in C$ distinct points. Let X be the blowup of $\overline{X} = C/p \sim q$. Fix odd theta characteristics $\eta, \mu \in S_C^-$ and denote by ξ' and ξ'' the double spin curves based on X restricting to (η, μ) on C (see Example 1.32).

Proposition 3.12. *One of the spin curves $\xi', \xi'' \in \Delta_0^{bb}$ belongs to $\widetilde{\Omega}_g$ if and only if (p, q) is a theta marking associated to (η, μ). Moreover, precisely one of ξ' and ξ'' will be in $\widetilde{\Omega}_g$ if the theta marking (p, q) is non-degenerate.*

Proof. Let $T \subset C \times C \setminus \Delta_C$ be the locus of distinct pairs of points such that $(X, L_\eta, L_\mu) \in \widetilde{\Omega}_g$ iff $(p, q) \in T$. Let $T', T'' \subset C \times C \setminus \Delta_C$ be the loci of degenerate and non-degenerate theta markings associated to (η, μ) respectively. We wish to show that $T = T' \cup T''$. Degenerate theta markings pose a problem for the main argument and for this reason we will first show that they may be ignored.

Since $\widetilde{\Omega}_g$ is closed one can see that T must be closed. Clearly degenerate theta markings are in the closure of non-degenerate theta markings, that is, $T' \subset \overline{T''}$. Therefore, once we show $T'' \subset T$ we may conclude $T' \subset T$. For this reason, for the rest of this proof we will assume $(p, q) \in C \times C \setminus \{\Delta_C \cup T'\}$ and prove that $(p, q) \in T$ iff $(p, q) \in T''$. This will imply $T = T' \cup T''$.

For notational convenience identify $\eta^{\otimes 2}$ and $\mu^{\otimes 2}$ with ω_C. We will denote X by

$$C \cup_{\substack{p \sim 0 \\ q \sim \infty}} \mathbb{P}^1.$$

Let $\mathcal{N}$ be the line bundle on X defined by $(\omega_C, \mathcal{O}_{\mathbb{P}^1}(2); f_p, f_q)$ where $f_p : \omega_C|_p \xrightarrow{\sim} \mathcal{O}_{\mathbb{P}^1}(2)|_p$ and $f_q : \omega_C|_q \xrightarrow{\sim} \mathcal{O}_{\mathbb{P}^1}(2)|_q$ are the gluing data.

Any spin structure L_η on X restricting to η on C and squaring to $\mathcal{N}$ is isomorphic to the line bundle defined by $(\eta, \mathcal{O}^1_{\mathbb{P}}(1); g_p, g_q)$ such that $g_r^{\otimes 2} = f_r$ for $r \in \{p, q\}$. By definition of a limit double spin curve the second spin structure L_μ restricting to μ on C must also square to $\mathcal{N}$. Therefore, L_μ is given by $(\mu, \mathcal{O}_{\mathbb{P}^1}(1); h_p, h_q)$ where $h_p^{\otimes 2} = f_p$ and $h_q^{\otimes 2} = f_q$.

Let $\zeta_\eta \in \mathrm{H}^0(C, \eta)$ and $\zeta_\mu \in \mathrm{H}^0(C, \mu)$ be non-zero sections. Let $D_\eta \subset X$ be the divisor in $|L_\eta|$, similarly define $D_\mu \subset X$. Let $r_\eta = D_\eta \cap \mathbb{P}^1$ and $r_\mu = D_\mu \cap \mathbb{P}^1$. Since we assumed that C is theta general we have $\eta \cap \mu = \emptyset$. Therefore $D_\eta \cap D_\mu \neq \emptyset$ iff $r_\eta = r_\mu$. We will now describe exactly when this happens.

Let $\langle x, y \rangle = \mathrm{H}^0(\mathbb{P}^1, \mathcal{O}(1))$ so that $0 = [1 : 0] = Z(x)$ and $\infty = [0 : 1] = Z(y)$. Then there are constants $a, b, c, d \in k$ such that:

$$\begin{array}{ll} g_q(\zeta_\eta) = ax & g_p(\zeta_\eta) = by \\ h_q(\zeta_\mu) = cx & h_p(\zeta_\mu) = dy \end{array}$$

Clearly r_η is cut out by $ax + by$ and r_μ is cut out by $cx + dy$. Therefore $r_\eta = r_\mu$ iff $ad = bc$.

Since we assumed that $(p, q) \notin T'$, we have $p + q \not\leq \eta$ and $p + q \not\leq \mu$. Therefore, $(a, b), (c, d) \neq (0, 0)$. Moreover, we will now prove that $r_\eta = r_\mu$ implies that a, b, c, d are all non-zero. Observe that $a = 0$ iff $p \leq \eta$ and that this implies $p \not\leq \mu$ because C is assumed theta general. This forces $c \neq 0$. Moreover, $a = 0$ implies $b \neq 0$ since $p + q \not\leq \eta$.

Then $ad = 0 \neq bc$ so that $r_\eta \neq r_\mu$. This proves that if $r_\eta = r_\mu$ then $a \neq 0$ but, using a similar argument, one can also show that $b, c, d \neq 0$.

Recall here that there are precisely two distinct isomorphism classes of double spin structures on X pulling back to (η, μ). If (L_η, L_μ) is one then the other will be given by (L'_η, L_μ) where $L'_\eta = (\eta, \mathcal{O}_{\mathbb{P}^1}(1); g_p, -g_q)$. Let $D'_\eta \in |L'_\eta|$ and $r'_\eta = D'_\eta \cap \mathbb{P}^1$. Then $r'_\eta = r_\mu$ iff $-ad = bc$. Since none of a, b, c, d is zero, at most one of the equalities $ad = bc$ and $-ad = bc$ can occur. Therefore, to complete the proof it remains to show that $(ad)^2 = (bc)^2$ iff (p, q) is a theta marking.

Let $W = \langle \zeta_\eta^{\otimes 2}, \zeta_\mu^{\otimes 2} \rangle \subset \mathrm{H}^0(\omega_C)$ and $\rho_p : W \to \omega_C|_p$, $\rho_q : W \to \omega_C|_q$ be the restriction maps. We claim that $(ad)^2 = (bc)^2$ iff $\ker \rho_p = \ker \rho_q$. Indeed, for any $(u, v) \in k^{\oplus 2}$ we have $\rho_p(u\zeta_\eta^{\otimes 2} + v\zeta_\mu^{\otimes 2}) = ua^2 + vb^2$. Therefore $\ker \rho_p = \ker \rho_q$ iff the matrix $\left(\begin{smallmatrix} a^2 & c^2 \\ b^2 & d^2 \end{smallmatrix}\right)$ has rank 1, that is, $(ad)^2 = (bc)^2$.

Now we wish to show that $\ker \rho_p = \ker \rho_q$ iff (p, q) is a theta marking associated to (η, μ). Indeed, the pencil of canonical divisors $|W| \subset |\omega_C|$ is precisely the associated divisors of (η, μ) and $D \in |\ker \rho_p|$ iff $p \leq D$. Therefore, $\ker \rho_p = \ker \rho_q$ iff $\exists D \in |W|$ such that $p + q \leq D$. This is the definition of a theta marking. $\square$

Since the generic pair of points $(p, q) \in C^2$ will not be a theta marking, Proposition 3.12 implies that the generic element in Δ_0^{bb} has non-intersecting pair of contact divisors. Therefore $\Delta_0^{bb} \not\subset \widetilde{\Omega}_g$ and by the proof of Theorem 3.1 we may conclude $\overline{\Omega}_g \cap \Delta_0^{bb} = \widetilde{\Omega}_g \cap \Delta_0^{bb}$ in $\overline{\mathcal{U}}_g^{--}$.

Notation 3.13. Let $C \hookrightarrow \mathbb{P}^n$ be an embedded curve and $(p, q) \in C^2$. By $L_{p,q}$ we will denote the secant line of C in $\mathbb{P}^n$ passing through p and q, this is to be interpreted as the tangent line at p if $p = q$.

Let $C \hookrightarrow \mathbb{P}^{g-1}$ be a general canonical curve of genus $g \geq 3$ and let $\{H_i \mid i = 1, \dots, \binom{2^g}{2}\}$ be the theta hyperplanes of C. For each $i \neq j$ we will denote by Λ_{ij} the intersection $H_i \cap H_j$. The following statement is evident.

Lemma 3.14. *A pair of points $(p, q) \in C^2$ is associated to (H_1, H_2) iff $L_{p,q} \cap \Lambda_{ij} \neq \emptyset$.*

Using this geometric reformulation we will prove that a general theta marking determines its associated pair of theta characteristics. Before we do so, let us mention what *general theta marking* should mean.

Remark 3.15. Given $(C, \mu_1, \mu_2) \in \mathcal{S}_g^{--}$ with $h^0(\mu_1) = h^0(\mu_2) = 1$, we can construct the scheme $T = \{(D, p, q) \mid D \in [\mu_1, \mu_2], (p, q) \in C^2, p + q \leq D\}$ which is a finite cover of $\mathbb{P}^1 \simeq [\mu_1, \mu_2]$. If a statement S is true for a general point *in every component of* T then we will say S is true for a general theta marking. Equivalently, S is true for a general theta marking iff S is true for *every* $p, q \leq D$ for a general $D \in [\mu_1, \mu_2]$.

Proposition 3.16. *Let $C \hookrightarrow \mathbb{P}^{g-1}$ be a general canonical curve of genus $g \geq 3$, then a general theta marking (p, q) is associated to a unique pair of theta hyperplanes (H_1, H_2).*

Proof. By Lemma 3.14 we need only prove that for a general theta marking (p, q) the line $L_{p,q}$ intersects only one of $\Lambda_{i,j}$. The proof of this fact consists of two steps which we prove as separate lemmas.

The first step is to show that $\Lambda_{ij} = \Lambda_{kl}$ iff $\{i, j\} = \{k, l\}$. Suppose to the contrary that $\{i, j, k, l\}$ contains at least three distinct indices i, j, k. Then $\Lambda_{ij} \subset H_k$, or equivalently $\operatorname{codim} H_i \cap H_j \cap H_k = 2$. However, for C general this never happens by Lemma 3.17.

The second step is to show that, given the arrangement of all Λ_{ij}'s, only finitely many bisecants intersects more than one Λ_{ij}. Lemma 3.18 proves this in greater generality. Therefore, a general hyperplane containing Λ_{ij} will not contain these finitely many exceptional bisecant lines. Picking a general divisor D associated to (H_i, H_j) and any $p + q \leq D$ we may now be certain that $L_{p,q}$ intersects only Λ_{ij}. $\square$

Lemma 3.17. *Let $C \in \mathcal{M}_g$ be general with $g \geq 3$. If H_1, H_2, H_3 are three distinct theta hyperplanes of C then* $\operatorname{codim} H_1 \cap H_2 \cap H_3 = 3$.

Proof. Equivalently, we wish to prove that the three divisors $D_i := H_i \cap C$ are not co-linear in $|\omega_C|$. This is a closed condition in the moduli space of triplets of distinct odd theta characteristics $\overline{\mathcal{S}}_g^{(1,1,1)}$. By Example 4.25 we know that $\overline{\mathcal{S}}_g^{(1,1,1)}$ has two irreducible components: $\overline{\mathcal{S}}_g^{\text{asyz}}$ and $\overline{\mathcal{S}}_g^{\text{syzy}}$.

With that observation, it suffices to find a single point on each of these components whose theta hyperplanes are not co-linear. We do this by specializing to hyperelliptic curves.

Let C be hyperelliptic with $w_1, \dots, w_{2g+2}$ its Weierstrass points. Take $\eta_1 = w_1 + \dots + w_{g-1}$ and $\eta_2 = w_2 + \dots + w_g$. Finally we take $\eta_3' = w_3 + \dots + w_{g+1}$ and $\eta_3'' = w_{g+1} + \dots + w_{2g-1}$. By Lemma III.1.7 the triplet $(\eta_1, \eta_2, \eta_3')$ is syzygetic while $(\eta_1, \eta_2, \eta_3'')$ is asyzygetic. Therefore $(C, \eta_1, \eta_2, \eta_3') \in \overline{\mathcal{S}}_g^{\text{syzy}}$ and $(C, \eta_1, \eta_2, \eta_3'') \in \overline{\mathcal{S}}_g^{\text{asyz}}$.

Consider $C \to \mathbb{P}^{g-1}$ mapped to a rational normal curve via the canonical map. To each η there will correspond a hyperplane H in $\mathbb{P}^{g-1}$. We claim that, in each of the two cases above, the corresponding triplet of hyperplanes do not intersect in a codimension two locus. To tackle both cases simultaneously, let $(H_3, \eta_3) \in \{(H_3', \eta_3'), (H_3'', \eta_3'')\}$ and for each i let $u_i \in \mathbb{P}^{g-1}$ be the image of the Weierstrass point w_i. Notice that $\operatorname{codim} H_1 \cap H_2 \cap H_3 = 2$ would imply that $H_1 \cap H_2 \subset H_3$. But $H_1 \cap H_2 = \langle u_2, \dots, u_{g-1} \rangle$. Certainly this hyperplane is not in H_3, because H_3 does not intersect the curve in u_2 (or else w_2 would be in the support of η_3). $\square$

Let $C \hookrightarrow \mathbb{P}^{g-1}$ be a *general* canonical curve of genus $g \geq 3$. Pick *any* two distinct codimension 2 subspaces $\Lambda_1, \Lambda_2 \in \mathbb{P}^{g-1}$.

Lemma 3.18 (Jason Starr)**.** *There exists only finitely many tuples of points $(p, q) \in C^2$ such that the line $L_{p,q}$ intersects both Λ_1 and Λ_2 iff $\Lambda_1 \cap \Lambda_2 \cap C = \emptyset$.*

Proof. If $\Lambda_1 \cap \Lambda_2 \cap C \neq \emptyset$ then we may simply pick a point p in this intersection and vary q freely, all the lines $L_{p,q}$ will necessarily intersect both Λ_i's. So we now assume $\Lambda_1 \cap \Lambda_2 \cap C = \emptyset$ and prove that only finitely many $L_{p,q}$ intersect both Λ_i's.

Let $V = \mathrm{H}^0(C, \omega_C)$ so the canonical map is $C \hookrightarrow \mathbb{P}(V)$. The locus of hyperplanes containing Λ_i defines a 2 dimensional space $W_i \subset V$. Let $\mathfrak{b}_i$ be the base locus of W_i and note that $\mathfrak{b}_i$ is supported on $\Lambda_i \cap C$. By hypothesis, $\mathfrak{b}_1$ and $\mathfrak{b}_2$ have disjoint supports.

The vector space W_i is really the image of another vector space $W_i' \subset \mathrm{H}^0(C, \omega_C(-\mathfrak{b}_i))$. If we let $\Pi_i := \mathbb{P}(W_i') \simeq \mathbb{P}^1$ then the projection of C from Λ_i is identified with the natural map

$$\pi_i : C \to \Pi_i$$

obtained by the surjection $W_i' \twoheadrightarrow \omega_C(-\mathfrak{b}_i)$. In particular, $\pi_i^* \mathcal{O}_{\Pi_i}(1) \simeq \omega_C(-\mathfrak{b}_i)$. We will study the map $\pi := (\pi_1, \pi_2) : C \to \Pi_1 \times \Pi_2$.

For $p \in C$, $p \not\leq \mathfrak{b}_i$ the span $\langle p, \Lambda_i \rangle$ is a hyperplane in $\mathbb{P}(V)$. By construction of π_i, the pullback divisor $\pi_i^{-1}\pi_i(p)$ coincides with $\langle p, \Lambda_i \rangle \cdot C - \mathfrak{b}_i$. In particular, for $p, q \in C \setminus \mathrm{supp}(\mathfrak{b}_i)$ the line $L_{p,q}$ intersects Λ_i iff for some $t_i \in \Pi_i$ we have $p + q \leq \pi_i^{-1}(t_i)$. Observe that $\pi^{-1}(t_1, t_2) = \min(\pi_1^{-1}(t_1), \pi_2^{-1}(t_2))$ where the minimum is taken pointwise on the coefficients of the divisors. It follows that $\forall p, q \in C \setminus \mathrm{supp}(\mathfrak{b}_1 + \mathfrak{b}_2)$, $L_{p,q} \cap \Lambda_i \neq \emptyset$ for $i = 1, 2$ iff $p + q \leq \pi^{-1}(t_1, t_2)$ for some $(t_1, t_2) \in \Pi_1 \times \Pi_2$.

Let B be the normalization of the image of π and factor π as follows:

$$C \xrightarrow{\nu} B \xrightarrow{\rho} \Pi_1 \times \Pi_2.$$

We will denote by d the degree of ν and by (n_1, n_2) the bidegree of the image of B. The previous paragraph implies that there are infinitely many (p, q) such that $L_{p,q} \cap \Lambda_i \neq \emptyset$ for $i = 1, 2$ only if $d > 1$.

Assume $d > 1$ to derive a contradiction. Since C is general in moduli, this forces $d \geq \lceil \frac{g}{2} \rceil + 1$ and $B \simeq \mathbb{P}^1$. Let $\mathcal{L} = \nu^* \mathcal{O}_B(1)$ and note that $h^0(\mathcal{L}) \geq 2$. Moreover, since $\rho^* \operatorname{pr}_i^* \mathcal{O}_{\Pi_i}(1) \simeq \mathcal{O}_B(n_j)$ where $j \neq i$ we then get:

$$\mathcal{L}^{\otimes n_j} \simeq \nu^* \rho^* \operatorname{pr}_i^* \mathcal{O}_{\Pi_i}(1) \simeq \pi^* \mathcal{O}_{\Pi_i}(1) \simeq \omega_C(-\mathfrak{b}_i).$$

Note however that $h^1(\mathcal{L}^{\otimes 2}) = h^0(\omega_C \otimes \mathcal{L}^{\otimes -2}) = h^0(\mathcal{L}^{n_j - 2} + \mathfrak{b}_i)$. Therefore $h^1(\mathcal{L}^{\otimes 2}) \geq 1$ if $n_j \geq 2$. On a Gieseker–Petri general curve, a line bundle L satisfying $h^0(L) \geq 2$ must satisfy $h^1(L^{\otimes 2}) = 0$ and so we conclude $n_1 = n_2 = 1$.

Then $B \hookrightarrow \Pi_1 \times \Pi_2$ is the graph of an isomorphism $\varphi : \Pi_1 \xrightarrow{\sim} \Pi_2$. This implies that the following diagram commutes:

$$\begin{tikzcd} & C \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \Pi_1 \arrow[rr, "\varphi"] & & \Pi_2 \end{tikzcd}$$

This can only happen if $\mathfrak{b}_1 = \mathfrak{b}_2$ and $W_1' = W_2'$. Therefore $\Lambda_1 = \Lambda_2$, a contradiction. $\square$

3.3 Over the component $\Delta_0^{b=}$

So far we could analyze the fiber of $\overline{\Omega}_g \to \overline{\mathcal{M}}_g$ over a theta nodal curve by relying on the modular interpretation of $\widetilde{\Omega}_g$ since there was no difference between $\widetilde{\Omega}_g$ and $\overline{\Omega}_g$ at the points we considered. However, for the present section the situation is different as $\Delta_0^{b=}$ is contained in $\widetilde{\Omega}_g$, but certainly not in $\overline{\Omega}_g$. Therefore we have to study deformations of double spin curves in $\Delta_0^{b=}$ in a direction pointing away from the boundary to determine whether or not they could lie in $\overline{\Omega}_g$.

Set-up 3.19. Let $\overline{X}$ be an irreducible curve of genus g with a single node $n \in \overline{X}$. Let $\nu : C \to \overline{X}$ be the normalization map and $p, q \in C$ be the preimages of the node n. Let $X \to \overline{X}$ be the blow-up of $\overline{X}$ at the node. We write $X = C \cup_{\substack{p\sim 0\\ q\sim\infty}} \mathbb{P}^1$.

A general element of $\Delta_0^{b=}$ is of the following form. Let $(L_1, L_2, \alpha_1, \alpha_2)$ be a double spin structure on X such that $L_i|_C \simeq \eta$, where η is an odd theta characteristic on C that is independent of the index i. Since $L_i|_{\mathbb{P}^1} \simeq \mathcal{O}_{\mathbb{P}^1}(1)$ but $L_1 \not\simeq L_2$, what distinguishes L_1 from L_2 is the way η and $\mathcal{O}_{\mathbb{P}^1}(1)$ are glued together.

We will make the following assumptions, guaranteeing the data is sufficiently general. All of the following are satisfied automatically if C itself is general, but we will often want to work with special curves and so we make the following precise generality assumption on our data.

Set-up 3.20. Assume the tuple $(C, \eta, p, q) \in \mathcal{S}^-_{g,2}$ is general in the following sense: $h^0(\eta) = 1$, $\mathfrak{d} \in |\eta|$ is reduced, $p, q \not\leq \eta$.

Having picked such a general element $(X, L_1, L_2) \in \Delta_0^{b=}$, the main result of this subsection, and the most difficult of the analogous results in the entire chapter, is the following.

Proposition 3.21. *The double spin curve $(X, L_1, L_2) \in \Delta_0^{b=}$ lies in $\overline{\Omega}_g$ if and only if the pencil $|\eta(p + q)|$ has a base point.*

Proof. Let $(\pi : \mathcal{X} \to B, \mathcal{L}_1, \mathcal{L}_2)$ be the universal deformation of (X, L_1, L_2). We will denote the closed point of B by 0. We will identify the central fiber of $\mathcal{X}$ with X. Construct the relative contact divisor $\mathcal{D}_i$ for $\mathcal{L}_i$. Since $h^0(L_i) = 1$ we have $\mathcal{D}_i \cap X \in |L_i|$ where $\mathcal{D}_i \cap C = \mathfrak{d}$ and $\mathcal{D}_i \cap \mathbb{P}^1$ is a smooth point. Let us write $\mathfrak{d} = r_1 + \cdots + r_{g-2}$. Since $\mathfrak{d}$ is assumed reduced, $\mathcal{D}_i$ is the disjoint union of $g-1$ sections $\mathfrak{r}_j^{(i)} : B \to \mathcal{X}$, $j = 1, \ldots, g-1$ such that $\mathfrak{r}_j^{(i)}(0) = r_i$ for $i < g-1$ and $\mathfrak{r}_{g-1}^{(i)}(0) \in \mathbb{P}^1$.

We will continue to denote the pullback of $\Delta_0^{b=}$ to B by $\Delta_0^{b=}$. We know that $\pi(\mathcal{D}_1 \cap \mathcal{D}_2)$ contains $\Delta_0^{b=}$. So $(X, L_1, L_2) \in \overline{\Omega}_g$ iff $\pi(\mathcal{D}_1 \cap \mathcal{D}_2) \setminus \Delta_0^{b=} \neq \emptyset$.

Since the generic curve has no intersecting contact divisors the sections $\mathfrak{r}_j^{(1)}$ and $\mathfrak{r}_j^{(2)}$ will not coincide. Thus $\mathfrak{r}_j^{(1)} \cap \mathfrak{r}_j^{(2)}$ is a union of codimension 2 components, all of which are locally complete intersections. We also know that for every $j < g-1$ the intersection $\mathfrak{r}_j^{(1)} \cap \mathfrak{r}_j^{(2)}$ will contain a component isomorphic to the image of a section of $\Delta_0^{b=}$.

The point (X, L_1, L_2) lies in $\overline{\Omega}_g$ iff $\exists j$ such that the intersection $\mathfrak{r}_j^{(1)} \cap \mathfrak{r}_j^{(2)}$ contains another component besides the one over $\Delta_0^{b=}$. This can only happen iff the intersection $\mathfrak{r}_j^{(1)} \cap \mathfrak{r}_j^{(2)}$ is non-transversal at r_j.

For notational simplicity, fix one $j < g-1$ and set $\mathfrak{r}^{(i)} = \mathfrak{r}_j^{(i)}$, $r = r_j$. Let $n_i \in \Omega_{\mathcal{X}}|_r$ be the conormal direction of $\mathcal{D}_i$ at r. The intersection $\mathfrak{r}^{(1)} \cap \mathfrak{r}^{(2)}$ is transversal at r iff $\dim_k \langle n_1, n_2 \rangle = 2$. Computing these normal directions by working on the universal deformation $\mathcal{X}$ would be technically challenging, so we wish to simplify the problem by working with a subfamily of $\mathcal{X}$. Lemma 3.22 allows us to do precisely that.

To complete the proof we construct a particularly simple family $\mathcal{Y} \to T$ where we can explicitly work with the intersection $D_1 \cap D_2$. On this family we will show that $D_1 \cap D_2$ is transversal iff $\eta(p+q)$ has no base locus, see Lemma 3.24. Constructing this family and computing the conormal directions will take up all of Section 3.3.1. □

Let $T \hookrightarrow B$ be a closed subscheme with $\dim_k \Omega_T|_0 \geq 1$ and T intersecting $\Delta_0^{b=}$ transversally. Let $\mathcal{Y}$ denote the pullback of $\mathcal{X}$ to T:

$$\begin{array}{ccc} \mathcal{Y} & \longrightarrow & \mathcal{X} \\ \big\downarrow & \ulcorner & \big\downarrow \\ T & \longrightarrow & B \end{array}$$

Note in particular that $\dim_k \Omega_{\mathcal{Y}}|_r \geq 2$. Denote by D_i the pullback $\mathcal{D}_i \cap \mathcal{Y}$.

Lemma 3.22. *The intersection $D_1 \cap D_2$ is transversal in $\mathcal{Y}$ if and only if the intersection $\mathcal{D}_1 \cap \mathcal{D}_2$ is transversal in $\mathcal{X}$.*

Proof. Once again we fix a point $r \leq \eta$ on C. Let $\mathcal{N}^*_{\mathcal{D}_1 \cap \mathcal{D}_2/\mathcal{X}}$ and $\mathcal{N}^*_{D_1 \cap D_2/\mathcal{Y}}$ be the conormal sheaves of the intersections. The intersection in $\mathcal{X}$ is transversal iff $\mathcal{N}^*_{\mathcal{D}_1 \cap \mathcal{D}_2/\mathcal{X}}|_r$ is 2 dimensional. The intersection in $\mathcal{Y}$ is transversal iff $\mathcal{N}^*_{D_1 \cap D_2/\mathcal{Y}}|_r$ is 2 dimensional. Note that the dimension of these spaces can not be greater than 2 in any case.

It follows from the defining exact sequences of the conormal sheaves that the restriction below is surjective:

$$\mathcal{N}^*_{\mathcal{D}_1 \cap \mathcal{D}_2/\mathcal{X}}|_r \twoheadrightarrow \mathcal{N}^*_{D_1 \cap D_2/\mathcal{Y}}|_r.$$

This proves that if the intersection $D_1 \cap D_2$ is transversal in $\mathcal{Y}$ then $\mathcal{D}_1 \cap \mathcal{D}_2$ is transversal in $\mathcal{X}$.

Conversely, if $\mathcal{D}_1 \cap \mathcal{D}_2$ is transversal at r then we observe the following: The component of $\mathcal{D}_1 \cap \mathcal{D}_2$ lying above $\Delta_0^{b=}$ and passing through r is the image of a section of $\Delta_0^{b=}$, in particular this component is smooth. By transversality assumption, this component must be the only component passing through r (here we use the fact that the remaining sections of $\mathcal{D}_i$'s are away from r). Furthermore, $\mathcal{D}_1 \cap \mathcal{D}_2$ base changes to $D_1 \cap D_2$ as we discussed in Chapter 2. Therefore, the only component of $D_1 \cap D_2$ passing through r is simply the image of a section of $\Delta_0^{b=} \cap T$. Since the intersection $\Delta_0^{b=} \cap T$ is assumed to be transversal, the intersection $D_1 \cap D_2$ is transversal at r. □

3.3.1 An almost trivial deformation of theta characteristics

We will continue with the Set-ups 3.19 and 3.20. Let $\overline{\mathcal{X}} \to \overline{B}$ be the universal deformation of the stable curve $\overline{X}$. We can write $\overline{B} = \operatorname{Spec} k[[t_1, \ldots, t_{3g-3}]]$ such that the subscheme

$\{t_1 = 0\}$ cuts out the locus of locally trivial deformations of $\overline{X}$, i.e., the deformations of $\overline{X}$ for which the node persists.

The universal deformation of a single theta characteristic (X, L) can then be denoted by $(\mathcal{X} \to B, \mathcal{L})$ such that forgetting $\mathcal{L}$ and stabilizing $\mathcal{X}$ gives a map $B \to \overline{B}$. We can write $B = \operatorname{Spec} k[[\tau, t_2, \ldots, t_{3g-3}]]$ such that the morphism $B \to \overline{B}$ is given by $t_i \mapsto t_i$ for $i \geq 2$ and $t_1 \mapsto \tau^2$. This is discussed in [Cor89] and [Jar98]. In Chapter I.4 we show that the universal deformation of a pair of theta characteristics (X, L_1, L_2) can be constructed over the same family of semi-stable curves: $(\mathcal{X} \to B, \mathcal{L}_1, \mathcal{L}_2)$.

Remark 3.23. We may make the following identifications $\Omega_{\overline{B}}|_0 = \langle t_1, \ldots, t_{3g-3}\rangle$ and $\Omega_B|_0 = \langle \tau, t_2, \ldots, t_{3g-3}\rangle$. Although the subspace $\langle t_1\rangle$ in $\Omega_{\overline{B}}$ is canonical, the splitting $\langle t_1\rangle \oplus \langle t_2, \ldots t_{3g-3}\rangle$ is not. However, having constructed $(\mathcal{X} \to B, \mathcal{L})$ the kernel of $\Omega_{\overline{B}}|_0 \to \Omega_B|_0$ provides us with that splitting. We suspect this must be an artifact of the choice we make in fixing $\mathcal{L}^{\otimes 2}$, a modification of $\omega_{\mathcal{X}/B}$, which is not canonical.

Consider the tangent direction $v : \operatorname{Spec} k[\varepsilon] \to B$ given by $\tau \mapsto \varepsilon$ and $t_i \mapsto 0$. Then the composition $\operatorname{Spec} k[\varepsilon] \to B \to \overline{B}$ is the zero tangent direction since $t_1 \mapsto \varepsilon^2 = 0$.

Let $T = \operatorname{Spec} k[\varepsilon] \xrightarrow{v} B$ and $\mathcal{Y} \to T$ be the pullback of $\mathcal{X}$ to T. By the universal property of $\overline{B}$ this means that the stabilization of the family $\mathcal{Y}$ is the trivial deformation of $\overline{X}$. We will denote this trivial deformation by $\overline{X}_\varepsilon \to \operatorname{Spec} k[\varepsilon]$.

As discussed in the proof of Proposition 3.21 the intersection of the relative contact loci, $\mathcal{D}_1 \cap \mathcal{D}_2$, is supported away from the exceptional component of $X \hookrightarrow \mathcal{X}$. This means that we can work with the stable models of these families. In any case, we demonstrate that the process of stabilizing the underlying family and pushing forward the theta characteristics is reversible, see Chapter III.2.

Let $\overline{\mathcal{E}}_i$ be the pushforward of L_i via the stabilization map $X \to \overline{X}$. Similarly we define $\mathcal{E}_i$ to be the pushforward of $\mathcal{L}_i|_{\mathcal{Y}}$ via the stabilization map $\mathcal{Y} \to X_\varepsilon$. If $\nu : C \to \overline{X}$ is the normalization map then $\overline{\mathcal{E}_i} = \nu_* \eta$ so we pick a non-zero section $s \in \mathrm{H}^0(C, \eta) = \mathrm{H}^0(\overline{X}, \overline{\mathcal{E}_i})$, independent of i. Pick a generating section $\sigma_i \in \mathrm{H}^0(X_\varepsilon, \mathcal{E}_i)$ such that $\sigma_i|_X = s$ for each i. Note that σ_i cuts out $D_i = \mathcal{D}_i \cap \mathcal{Y}$. Therefore, the following proof is based upon an explicit determination of σ.

Lemma 3.24. *The relative contact divisors $D_1, D_2 \subset X_\varepsilon$ intersect transversally at every point $r \leq \eta$ iff $\eta(p+q)$ has no base points.*

Proof. For notational convenience, let us work at first with a single root. So we start with $(\overline{X}, \overline{\mathcal{E}})$ and its deformation $(\overline{X}_\varepsilon, \mathcal{E})$. We will describe sections of $\mathcal{E}$ explicitly.

We define $\overline{U} = \overline{X} \setminus \{n, r_1, \ldots, r_{g-2}\}$. The locus $\overline{U}$ will also be identified with $C \setminus \{p, q, r_1, \ldots, r_{g-2}\}$. We trivialize $\eta|_{\overline{U}}$ via the section $s|_{\overline{U}}$. Let $\overline{V}_i = \operatorname{Spec} \hat{\mathcal{O}}_{C, r_i} \to C$ be the formal neighbourhood of r_i. Fix a trivialization of η on $\mathcal{O}_{C, r_i}$, which necessarily identifies s with a uniformizer x_i of the local ring. We then make the identification $\overline{V}_i = \operatorname{Spec} k[[x_i]]$. Finally let $\overline{W} = \operatorname{Spec} \hat{\mathcal{O}}_{\overline{X}, n} \to \overline{X}$ be the formal neighbourhood of the node. Pick uniformizers $x \in \mathfrak{m}_p \subset \mathcal{O}_{C,p}$ and $y \in \mathfrak{m}_q \subset \mathcal{O}_{C,q}$, then we may make the identification $\overline{W} = \operatorname{Spec} k[[x, y]]/(xy)$.

On $\overline{W}$ we define the following module: $\overline{E} = \langle \overline{e}_1, \overline{e}_2 \mid y\overline{e}_1 = x\overline{e}_2 = 0\rangle$. It is easy to see that $\overline{\mathcal{E}}|_{\overline{W}} \simeq \overline{E}$ — this is discussed thoroughly in [Jar98] and in Part I of this thesis.

The torsion-free root $\overline{\mathcal{E}}$ is obtained by gluing the trivial line bundles on $\overline{U}$ and $\overline{V}_i$'s to $\overline{E}$ on $\overline{W}$. We see that s corresponds to the collection of sections $((\overline{U}, 1), (\overline{V}_i, x_i), (\overline{W}, \overline{e}_1 + \overline{e}_2))$ on these trivializations.

As we will use them later for comparison, let us describe the gluing data for $\overline{\mathcal{E}}$. Let $\overline{V}_i^o = \overline{U} \times_C \overline{V}_i$ and $\overline{W}^o = \overline{U} \times_{\overline{X}} \overline{W}$. We can then describe the gluing data $g_{\overline{V}_i} : \mathcal{O}_{\overline{U}}|_{\overline{V}_i^o} \to \mathcal{O}_{\overline{V}_i}|_{\overline{V}_i^o}$ as $1 \mapsto x_i$ and $g_{\overline{W}} : \mathcal{O}_{\overline{U}}|_{\overline{W}^o} \to \overline{E}|_{\overline{W}^o}$ as $1 \mapsto (\overline{e}_1 + \overline{e}_2)|_{\overline{W}^o}$. Note that $\overline{E}|_{\overline{W}^o} = \langle \overline{e}_1 \rangle \oplus \langle \overline{e}_2 \rangle \simeq k((x)) \times k((y))$.

The underlying topological spaces of $\overline{X}$ and of $\overline{X}_\varepsilon$ are identified via the inclusion $\overline{X} \hookrightarrow \overline{X}_\varepsilon$ and the scheme structure of $\overline{X}_\varepsilon$ is the trivial thickening of $\overline{X}$. We will use the same letters $U, V_i, W \to \overline{X}_\varepsilon$ to denote the pullbacks of the corresponding morphisms defined above for $\overline{X}$. The underlying scheme structures change in the obvious way. For instance, the pullback W of $\overline{W}$ is identified with $\operatorname{Spec} k[\varepsilon][[x, y]]/(xy)$.

Away from the node $\overline{\mathcal{E}}$ is locally trivial. We show in Lemma I.3.9 that locally trivial roots deform uniquely up to unique isomorphism. This means that the trivial deformation of $\overline{\mathcal{E}}$ and the deformation $\mathcal{E}$ must be uniquely isomorphic away from the node (in a way compatible with the squaring maps, which we omit from notation). Therefore, the gluing data $g_{\overline{V}_i}$ and $g_{\overline{W}}$ pullback to give the gluing data g_{V_i} and g_W of $\mathcal{E}$. There are no surprises for g_{V_i}, we can still write it as $1 \mapsto x_i$, however the pullback of g_W will require greater care to describe.

Let $E := E(\varepsilon, \varepsilon) = \langle e_1, e_2 \mid ye_1 - \varepsilon e_2 = xe_2 - \varepsilon e_1 = 0\rangle$ be defined over $k[\varepsilon][[x, y]]/(xy)$. We know that $\mathcal{E}|_W \simeq E$, see [Jar98] or Part I of this thesis. Let us view the open set $W^o = W \times_{\overline{X}_\varepsilon} U \subset W$ as the disjoint union of the open sets W_x and W_y of W, obtained by inverting x and y respectively. Let $E_x := E|_{W_x}$ and similarly define E_y. Note that $E_x = \langle e_1 \rangle \simeq k[\varepsilon]((x))$ (where we map e_1 to 1) and similarly $E_y \simeq k[\varepsilon]((y))$. The restriction map $E \to E_x \simeq \langle e_1 \rangle$ sends $e_1 \mapsto e_1$ and $e_2 \mapsto \frac{\varepsilon}{x} e_1$. Then gluing data $g_{\overline{W}}$ computed above pullsback to $\mathcal{O}_U|_{W^o} \to E_x \oplus E_y \simeq k[\varepsilon]((x)) \oplus k[\varepsilon]((y)) : 1 \mapsto (1, 1)$. We wish to emphasize that the last expression is different from $(e_1 + e_2)|_{W^o} = (1 + \frac{\varepsilon}{x}, 1 + \frac{\varepsilon}{y})$.

Pick a section $\sigma \in \mathrm{H}^0(\mathcal{E})$ such that $\sigma|_X = s$. We may denote σ by a tuple $((U, \sigma_U), (V_i, \sigma_{V_i}), (W, \sigma_W))$ where $\sigma_U \in \mathcal{O}_U$, $\sigma_{V_i} \in k[\varepsilon][[x_i]]$ and $\sigma_W \in k[\varepsilon][[x, y]]/(xy)$. Since $\sigma|_X = s$, we may in fact write:

$$\begin{aligned} \sigma_U &= 1 + a\varepsilon \\ \sigma_{V_i} &= x_i + a_i\varepsilon \\ \sigma_W &= (1 + \varepsilon u)e_1 + (1 + \varepsilon v)e_2 \end{aligned}$$

where $a \in \mathrm{H}^0(\mathcal{O}_C|_U)$, $a_i \in k[[x_i]]$ and $u \in k[[x]]$, $v \in k[[y]]$.

Now $\sigma_W|_{W^o} = (1 + \varepsilon(\frac{1}{x} + u), 1 + \varepsilon(\frac{1}{y} + v))$. Since σ_U and σ_W must agree on W^o we conclude that a, viewed as meromorphic function on C, must have a simple pole on p and q. Checking compatibility on V_i^o we conclude that $x_i a = a_i \in k[[x_i]]$. This means that a has at most a simple pole at r_i. The function a is regular everywhere else.

Let $\mathfrak{d} \in |\eta|$ and $\mathfrak{d}_\infty = \mathfrak{d} + p + q \in |\eta(p + q)|$. Our analysis of a implies that $a \in L(\mathfrak{d}_\infty) = \{f \in K(C) \mid (f) + \mathfrak{d}_\infty \geq 0\} \simeq \mathrm{H}^0(\eta(p + q))$ and, because a must have poles at p and q, the function a can not be constant. Since $h^0(\eta(p+q)) = 2$ the vector space $L(\mathfrak{d}_\infty)$ is generated by 1 and a. We will now show that any $a' \in L(\mathfrak{d}_\infty)$ can be obtained from a section $\sigma' \in \mathrm{H}^0(\varepsilon)$. Indeed, scaling our σ by a constant $c_0 + c_1\varepsilon \in k[\varepsilon]$ we change a to $c_0 a + c_1$.

Suppose r is a base point of $|\eta(p+q)|$. We use the hypothesis $p, q \not\leq \eta$ and apply Riemann–Roch to see that $r \leq \eta$. Therefore, a has poles precisely on $\mathfrak{d}_\infty = \mathfrak{d} + p + q$ iff $\eta(p+q)$ has no base points. In particular, $\forall i$ the function $a_i = x_i a$ is non-zero at r_i iff $\eta(p+q)$ has no base points.

Let D be the divisor in $\overline{X}_\varepsilon$ cut out by σ. We wish to compute the conormal of D at $r_i \leq \eta$. We will do so in using the formal neighbourhood V_i of r_i. Note that $\Omega_{V_i}|_{r_i} = \langle \mathrm{d}x_i, \mathrm{d}\varepsilon \rangle$. Since we know σ can locally be described by the function $x_i + a_i\varepsilon \in k[\varepsilon][[x_i]]$ the conormal of D at r_i is $\mathrm{d}(x_i + a_i\varepsilon)|_{r_i} = \mathrm{d}x_i + a_i(0)\mathrm{d}\varepsilon$.

Finally, we are ready to compare the conormals of D_1 and D_2. For each $i = 1, 2$ let $\sigma_i \in \mathrm{H}^0(\mathcal{E}_i)$ be a generating global section restricting to s. We will apply the argument to each $\mathcal{E}_i$. There is one final twist here: if we choose a trivialization $\mathcal{E}_1|_W \simeq E$ then the same gluing data work for $\mathcal{E}_2$ only if we make the identification $\mathcal{E}_2|_W \simeq E'$ where $E' = E(-\varepsilon, -\varepsilon) = \langle e_1, e_2 \mid ye_1 + \varepsilon e_2 = xe_2 + \varepsilon e_1 = 0 \rangle$. This is the main premise of the notion of synchronization of roots introduced in Part I.

It is easy to check that if $\sigma_{1,U} = 1 + a\varepsilon$ then $\sigma_{2,U} = 1 - a\varepsilon$. In particular, with $a_i = x_i a \in k[[x_i]]$ as above, if we denote the normal direction of D_j at r_i by n_{j,r_i} then we have:

$$\begin{aligned} n_{1,r_i} &= \mathrm{d}x_i + a_i(0)\mathrm{d}\varepsilon \\ n_{2,r_i} &= \mathrm{d}x_i - a_i(0)\mathrm{d}\varepsilon \end{aligned}$$

Therefore, D_1 and D_2 intersect transversely at r_i iff $a_i(0) \neq 0$. We discussed above that $a_i(0) \neq 0$ for all i precisely when $\eta(p+q)$ has no base points. □

In light of this result, we need to study the points $p, q \in C$ for which the twisted theta characteristic $\eta(p+q)$ has a base point to say more about the fibers of $\overline{\Omega}_g \cap \Delta_0^{b=}$ over $\Delta_0^{b=}$.

3.3.2 Base loci of twisted theta characteristics

For a general element $(C, \eta, p, q) \in \mathcal{S}_{g,2}^-$ one can easily show that the twisted theta characteristic $\eta(p+q)$ will not have a base locus: use the fact that $\mathcal{S}_{g,2}^-$ is irreducible, degenerate to a hyperelliptic curve and use Lemma 3.25 below.

Lemma 3.25. *Let C be a hyperelliptic curve of genus $g \geq 2$ and let η be an odd theta characteristic on C with $h^0(\eta) = 1$. For $(p, q) \in C^2$ the pencil $|\eta(p+q)|$ has base locus iff one of the following hold: $p \leq \eta$, $q \leq \eta$ or $p + q \equiv g_2^1$.*

Proof. The line bundle $\eta(p+q)$ has base locus containing $r \in C$ iff $h^0(\eta + p + q - r) = 2$, or equivalently, $h^0(\eta + r - p - q) = 1$. We first assume $r \not\leq \eta$, then $h^0(\eta + r) = 1$ since $h^0(\eta - r) = 0$. So $p + q \leq \eta + r$ forces $p \leq \eta$ or $q \leq \eta$. Next, we assume $r \leq \eta$. Labeling the

Weierstrass points of C appropriately we may assume $\eta \equiv w_1 + \cdots + w_{g-1}$ and $r = w_{g-1}$. Then $\eta + r \equiv w_1 + \cdots + w_{g-2} + g^1_2$. Now if $p, q \leq \eta$ then $h^0(\eta + r - p - q)) = 1$ and $\eta(p+q)$ has a base point. If $p, q \not\leq \eta$ then $h^0(\eta + r - p - q) = 1$ iff $p + q \equiv g^1_2$. $\square$

Corollary 3.26. *The relative contact divisors $\mathcal{D}_1, \mathcal{D}_2 \subset \mathcal{C} \to \overline{\mathcal{U}}_g^{--}$ intersect generically transversally over the boundary component $\Delta_0^{b=}$.*

Proof. The condition regarding base loci in Proposition 3.21 checks for this transversality. Since we showed that in general $\eta(p+q)$ has no base locus, the intersection is transversal over $\Delta_0^{b=}$. $\square$

However, we need to show that $\eta(p+q)$ has no base locus when (p, q) is a general theta marking in order to conclude $f_{\overline{X}}^{b=}$ from Section 3.1 equals 0. Let us start with a simple lemma.

Pick $1 \leq n \leq g-1$ and consider the stack $\mathcal{T}_g^{(n)} = \{(C, \eta, r_1, \ldots, r_n) \in \mathcal{S}_{g,n}^- \mid h^0(\eta) = 1,\ \eta \text{ is reduced},\ r_1 + \cdots + r_n \leq \eta\}$.

Lemma 3.27. *The stack $\mathcal{T}_g^{(n)}$ is irreducible.*

Proof. Since the moduli space $\mathcal{S}_g^-$ is irreducible, $\mathcal{T}_g^{(n)}$ is irreducible iff the monodromy action of the map $\mathcal{T}_g^{(n)} \to \mathcal{S}_g^-$ is transitive on the fibers. Take a hyperelliptic curve C with an odd theta characteristic $\eta = w_1 + \cdots + w_{g-1}$. It is well known that the monodromy action of the space of hyperelliptic curves on the Weierstrass points of the curves is the full symmetric group Σ_{2g+2}. In particular the subgroup $\Sigma_{g-1} \subset \Sigma_{2g+2}$ fixes (C, η) and therefore lifts to a transitive monodromy action of $\mathcal{T}_g^{(n)} \to \mathcal{S}_g^-$ over (C, η). $\square$

Remark 3.28. Monodromy arguments such as the one above can be completely formalized in the algebraic setting for base fields other than $\mathbb{C}$. This is particularly easy for Weierstrass points of curves. For example, for the irreducibility of $\mathcal{T}_g^{(n)}$ one simply constructs a surjective map from the moduli space of marked genus 0 curves to $\mathcal{T}_g^{(n)}$. However, we find the language of monodromy far more intuitive and we will continue to use it without further remark.

Proposition 3.29. *Let $g \geq 3$. For sufficiently general $C \in \mathcal{M}_g$ and any triplet of odd theta characteristics η, μ_1, μ_2 on C with $\mu_1 \not\cong \mu_2$, if we take a general theta marking $(p, q) \in C^2$ associated to (μ_1, μ_2) then the pencil $|\eta(p+q)|$ has no base locus.*

Proof. Let $\mathcal{S} = \{(C, \mu_1, \mu_2)\}$ be the moduli space of distinct rigid odd spin curves of genus g. Over $\mathcal{S}$ we may construct the $\mathbb{P}^1$-bundle of associated divisors: $\mathcal{P} = \{(C, \mu_1, \mu_2, D \in [\mu_1, \mu_2])\}$. Since $\mathcal{S}$ is irreducible, so is $\mathcal{P}$. Fix a generic element $(C, \mu_1, \mu_2, D) \in \mathcal{P}$. It will be sufficient to show that $\forall p + q \leq D$ and $\forall \eta \in S_C^-$ the line bundle $\eta(p+q)$ has no base locus.

First of all, only finitely many associated divisors $D \in [\mu_1, \mu_2]$ will have common support with an odd theta characteristic. We may thus assume that $\forall \eta \in S_C^-$ that $D \cap \eta = \emptyset$. If $\eta(p+q)$ has a base point r for some $p + q \leq D$ then, since $p, q \not\leq \eta$, Riemann–Roch implies that $r \leq \eta$ and $p + q \leq \eta + r$.

In other words, assuming D is reduced, the existence of $p+q \leq D$ such that $\eta(p+q)$ has base points is equivalent to the existence of $r \leq \eta = r_1 + \cdots + r_{g-1}$ and $E \in |\eta + r|$ such that $\#D \cap E \geq 2$. We will show that for D general, this last condition will not occur.

Fix $r \leq \eta$ and let $C \to |\eta + r| : p \mapsto E_p$ where $E_p \in |\eta + r| \simeq \mathbb{P}^1$ is the only divisor with $p \leq E_p$. This morphism has degree g provided it has no base points, which we now prove. Clearly $r' \leq \eta + r$ is a base point only if $r' \leq \eta$. Therefore, we need only consider the moduli space $\mathcal{T} = \{(C, \eta, r, r') \mid r + r' \leq \eta\}$ which is irreducible by Lemma 3.27. Considering $h^0(\eta + r - r')$, we see that the locus where r' is a base point of $\eta + r$ is closed in $\mathcal{T}$. In particular, since $\mathcal{T}$ is finite over $\mathcal{S}_g^-$, for a generic (C, η) and any $r \leq \eta$ either the line bundle $\eta + r$ has no base points or every $r' \leq \eta - r$ is a base point. The latter implies $h^0(2r) = 2$ which is impossible since C is generic.

Let $C \to [\mu_1, \mu_2] : p \mapsto D_p$ be defined so that D_p is the unique associated divisor having $p \leq D_p$. This morphism has degree $2g - 2$. Using both of these maps we get $\varphi : C \to |\eta + r| \times [\mu_1, \mu_2] \simeq \mathbb{P}^1 \times \mathbb{P}^1$. The map φ is generically injective precisely when for the generic $D \in [\mu_1, \mu_2]$ and any $E \in |\eta + r|$ we have $\#D \cap E \leq 1$.

We now argue as in Lemma 3.18. Suppose that φ is not generically injective. Then, C being general, the geometric genus of $\operatorname{im}\varphi$ must be 0. Let $\nu : \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$ be the normalization of the image of φ. Then φ factors as $\nu \circ s$ for some $s : C \to \mathbb{P}^1$.

Suppose $\operatorname{im}\varphi$ is a curve of bidegree (a, b). Then $\nu^*\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(0,1) \simeq \mathcal{O}_{\mathbb{P}^1}(a)$ and $\nu^*\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,0) \simeq \mathcal{O}_{\mathbb{P}^1}(b)$. On the other hand, we have $\varphi^*\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(0,1) \simeq \omega_C$ and $\varphi^*\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,0) \simeq \eta + r$. In particular, $s^*\mathcal{O}_{\mathbb{P}^1}(b) \simeq \eta + r$. However, $\eta + r$ will not have an effective root so $b = 1$. Since $s^*\mathcal{O}_{\mathbb{P}^1}(1) \simeq \eta + r$ the degree of the map s is g.

On the other hand, $s^*\mathcal{O}_{\mathbb{P}^1}(a) \simeq \omega_C$ and taking degrees this gives $ag = 2g - 2$. But when $g \geq 3$, g does not divide $2g - 2$, a contradiction. $\square$

3.4 Smooth and singular spin curves

Now we will study the behavior of $\overline{\Omega}_g$ near Δ_0^{bn}. Let $(\overline{X}, \mathcal{E}_1, \mathcal{E}_2) \in \Delta_0^{bn}$ be such that $\overline{X} = C/(p \sim q)$ has a single node and the theta characteristics $\mathcal{E}_i$ are rigid. The first theta characteristic $\mathcal{E}_1$ is obtained by pushing forward an odd theta characteristic $\eta \in S_C^-$ and the second $\mathcal{E}_2$ is obtained by gluing a root $\tau \in \sqrt{\omega_C(p+q)}$. See Example 1.28 for more details. The space Δ_0^{nb} has the order of these two types of spin structures reversed, otherwise its treatment is identical.

We showed in Section 2.2 that if $(X, L_1, L_1) \in \Delta_0^{bn}$ has a common contact point then this is because the supports of η and τ intersect on C.

Lemma 3.30. *For any $(C, p, q) \in \mathcal{M}_{g,2}$ and $\tau \in \sqrt{\omega_C(p+q)}$ if $h^0(\tau) = 1$ then $p, q \not\leq \tau$. Moreover, for general $(C, p, q) \in \mathcal{M}_{g,2}$ and for every $\tau \in \sqrt{\omega_C(p+q)}$ we have $h^0(\tau) = 1$.*

Proof. The first sentence is due to Cornalba [Cor89], we prove it here for convenience. For the second sentence we use Theorem 1.1 of Polishchuk [Pol06]. Take $(C, p, q) \in \mathcal{M}_{g,2}$ general and pick any $\tau \in \sqrt{\omega_C(p+q)}$. By Riemann–Roch we have

$$h^0(\tau) - h^0(\tau - p - q) = 1.$$

In particular, $\exists \sigma \in \mathrm{H}^0(\tau) \setminus \mathrm{H}^0(\tau - p - q)$ and such a section will invariably satisfy $\sigma^{\otimes 2} \in \mathrm{H}^0(\omega_C(p+q))$ so that $\operatorname{res}_p \sigma^{\otimes 2} = -\operatorname{res}_q \sigma^{\otimes 2}$. Therefore, $\sigma|_p = 0$ iff $\sigma|_q = 0$. Since we picked σ so that it does not vanish on both p and q, σ does not vanish on either p or q. This proves that if $h^0(\tau) = 1$ then $p, q \not\leq \tau$.

Note that $h^0(\tau) = 1$ iff $h^0(\tau - p - q) = 0$. But $h^0(\tau - p - q) \neq 0$ for some $\tau \in \sqrt{\omega_C(p+q)}$ iff $\exists D' \geq 0$ such that $2D' + p + q \in |\omega|$. Let $Y \subset \mathcal{M}_{g,2}$ be the locus of tuples (C, p, q) for which there exists such D'. Theorem 1.1 of [Pol06] implies that Y is closed and with at most divisorial components. Hence for general $(C, p, q) \in \mathcal{M}_{g,2}$ there is no such D'. $\square$

Lemma 3.31. *Let C be a hyperelliptic curve of genus $g \geq 1$ and let $p, q \in C$ be non-conjugate points, i.e., $p \neq \bar{q}$. Then $\forall \tau \in \sqrt{\omega_C(p+q)}$ we have $h^0(\tau) = 1$ and $p, q \not\leq \tau$. Moreover, for any Weierstrass point $w \in C$ we have $w \not\leq \tau$.*

Proof. Choose $D \in |\tau|$ and notice that $2D + \bar{p} + \bar{q} \in |(g+1)\mathfrak{g}^1_2|$ where $\mathfrak{g}^1_2$ is the degree 2 pencil on C. Let $e \in |\mathfrak{g}^1_2|$ be any divisor and suppose that $e \leq 2D + \bar{p} + \bar{q}$. Then the effective divisor $E = 2D + \bar{p} + \bar{q} - e$ lies in the linear system $|g\mathfrak{g}^1_2|$. Using Riemann–Roch one can see that all divisors in $|g\mathfrak{g}^1_2|$ are invariant under conjugation. This implies that the coefficients of any two conjugate points appearing in E must be equal. However, it is clear that the coefficient of $\bar{p}$ must be odd whereas the coefficient of p must be even. This is a contradiction, implying that $\forall e \in |\mathfrak{g}^1_2|$ we have $e \not\leq 2D + \bar{p} + \bar{q}$.

Taking $e = 2w$ for a Weierstrass point $w \in C$ implies $w \not\leq D$. Taking $e = p + \bar{p}$ and then $q + \bar{q}$ implies that $p, q \not\leq D$. That is $h^0(\tau - p - q) = 0$. Then the proof of Lemma 3.30 implies that $h^0(\tau) = 1$. $\square$

Lemma 3.32. *If $(C, p, q) \in \mathcal{M}_{g,2}$ is generic then for all $\eta \in S^-_C$ and for all $\tau \in \sqrt{\omega_C(p+q)}$ we have $\eta \cap \tau = \emptyset$.*

Proof. Let $U \subset \mathcal{M}_{g,2}$ be a non-empty open substack such that for all $(C, p, q) \in U$, $\forall \tau \in \sqrt{\omega_C(p+q)}$ we have $h^0(\tau) = 1$ (see Lemma 3.30). Over U we construct the stack $\mathcal{S} \to U$ given by

$$\mathcal{S} = \{(C, p, q, \eta) \mid \eta \in S^-_C,\ h^0(\eta) = 1\}.$$

Note that $\mathcal{S}$ is irreducible. We define the locus $T \subset \mathcal{S}$ such that it parametrizes tuples $(C, p, q, \eta) \in \mathcal{S}$ for which $\exists \tau \in \sqrt{\omega_C(p+q)}$ satisfying $\tau \cap \eta \neq \emptyset$. Clearly, the locus T is a closed substack.

Since $\mathcal{S} \to U$ is quasi-finite, we are done once we show that T does not equal $\mathcal{S}$. To show this, we need only construct a single example in the complement of T.

Let C be a hyperelliptic curve and $\eta = w_1 + \cdots + w_{g-1}$ an odd theta characteristic on C. We let p and q be non-conjugate points in C. By Lemma 3.31 we have $\forall \tau \in \sqrt{\omega_C(p+q)}$, $h^0(\tau) = 1$ and $\eta \cap \tau = \emptyset$. So $(C, p, q) \in U$ and $(C, p, q, \eta, r) \in \mathcal{S} \setminus T$. $\square$

Remark 3.33. Lemma 3.32 implies that Δ^{bn}_0 and Δ^{nb}_0 are not contained in $\widetilde{\Omega}_g$. Therefore, $\overline{\Omega}_g \cap \Delta^{bn}_0 = \widetilde{\Omega}_g \cap \Delta^{nb}_0$ in $\overline{\mathcal{U}}^{--}_g$.

Now we will prove stronger versions of the Lemmas 3.30 and 3.32. Despite this, we proved Lemmas 3.30 and 3.32 because we view those two facts as fundamental and deserving their own proof. The results we now prove maybe stronger, but the proofs are harder (not least because they reference Theorem III.1.8) and they rely on objects very specific to our own inquiry.

Lemma 3.34. *Pick $(C, p, q) \in \mathcal{M}_{g,2}$, where C is general and (p, q) is a general theta marking. Then $\forall \tau \in \sqrt{\omega_C(p+q)}$ we have $h^0(\tau) = 1$.*

Proof. The moduli space $Z = \{(C, \mu_1, \mu_2, D \in [\mu_1, \mu_2], p, q) \mid p + q \leq D\}$ is irreducible (Theorem III.1.8). Moreover, on Z the condition that all the roots $\tau \in \sqrt{\omega_C(p+q)}$ have one dimensional sections is open. Therefore, we need only check this condition on a single point of Z. We do this by specializing to hyperelliptic curves and applying Lemma 3.31. It is clear that we may take non-conjugate points as theta markings. □

Definition 3.35. Let C be a hyperelliptic curve and $(p, q) \in C^2$ a theta marking. We will call (p, q) a *theta marking of type 1* if $p = \bar{q}$ and *of type 2* otherwise.

Let us define $\tilde{\mathcal{S}} = \{(C, \eta, \mu_1, \mu_2)\}$ to be the moduli space of genus g curves with 3 rigid odd spin structures, with the condition that $\mu_1 \not\simeq \mu_2$. Let $\tilde{\mathcal{P}} \to \tilde{\mathcal{S}}$ be the $\mathbb{P}^1$-bundle over $\tilde{\mathcal{S}}$ parametrizing, in addition, the associated divisors $D \in [\mu_1, \mu_2]$. Finally let $\tilde{Z} = \{(C, \eta, \mu_1, \mu_2, D, p, q) \mid p + q \leq D\} \to \mathcal{P}$ parametrize the theta markings. All three of these moduli spaces obtained by forgetting η are denoted by $\mathcal{S}, \mathcal{P}$ and Z respectively.

Lemma 3.36. *In the moduli space $\tilde{Z}$, every component contains hyperelliptic curves with type 2 theta markings.*

Proof. Pick a general $(C, \mu_1, \mu_2) \in \mathcal{S}$. The proof of Theorem III.1.8 makes use of a monodromy between the curves $C \to [\mu_1, \mu_2] \simeq \mathbb{P}^1$ to connect type 1 and type 2 theta markings on hyperelliptic curves. Since this monodromy argument takes place over a single fiber of the morphism $\mathcal{P} \to \mathcal{S}$, we may replicate that argument here using $\tilde{\mathcal{P}} \to \tilde{\mathcal{S}}$ to connect a given type 1 point in $\tilde{Z}$ to some type 2 point in $\tilde{Z}$. □

Proposition 3.37. *Pick a general curve $C \in \mathcal{M}_g$ and a general theta marking $(p, q) \in C^2$. Then $\forall \eta \in S_C^-$ and $\forall \tau \in \sqrt{\omega_C(p+q)}$ we have $\eta \cap \tau = \emptyset$.*

Proof. Since the forgetful map $\tilde{Z} \to Z$ is quasi-finite, if we show that on each component of $\tilde{Z}$ the desired condition is true on a non-empty open set, we will be done. The advantage of working with $\tilde{Z}$ is that we get to check this condition for a single theta characteristic, rather than all theta characteristics on a given curve. This is especially useful on hyperelliptic curves where only some of the odd theta characteristics are rigid.

For $(C, \eta, \mu_1, \mu_2, D, p, q) \in \tilde{Z}$, the condition that $\forall \tau \in \sqrt{\omega_C(p+q)}$ we have $h^0(\tau) = 1$ and $\tau \cap \eta = \emptyset$ is open. Therefore it suffices to exhibit a single example where this statement holds on each component of $\tilde{Z}$. We will do this by specializing to hyperelliptic curves with theta markings of type 2. In light of Lemma 3.36 this provides us with enough examples.

Now assume $(C, \eta, \mu_1, \mu_2, D, p, q) \in \tilde{Z}$ is such that C is hyperelliptic, all theta characteristics are rigid and $p \neq \bar{q}$. By Lemma 3.31 we know that $\forall \tau \in \sqrt{\omega_C(p+q)}$ we have $h^0(\tau) = 1$ and, since η is a sum of Weierstrass points, $\tau \cap \eta = \emptyset$. □

3.5 Smooth spin structures

Let $(\overline{X}, L_1, L_2) \in \Delta_0^{nn}$ be a contact general curve, that is $h^0(L_i) = 1$ and $\overline{X}$ has a single node. The spin structures L_i are smooth by definition of Δ_0^{nn}, so that L_i are line bundles on the stable spin curve $\overline{X}$. We showed in 2.2 that the limit contact divisor of L_i is the unique divisor $D_i \in |L_i|$. Moreover, Lemma 3.30 implies that D_i is supported away from the node.

With $\nu : C \to \overline{X}$ the normalization and $p, q \in C$ the preimages of the node, we will have $\tau_i = L_i|_C \in \sqrt{\omega_C(p+q)}$. Therefore, $D_1 \cap D_2 = \emptyset$ on $\overline{X}$ iff $\tau_1 \cap \tau_2 = \emptyset$ on C.

Recall from Section 1.7 that the clutching maps for $\Delta_0^n \subset \overline{\mathcal{S}}_g^-$ were defined using the moduli space $T_{g,2} = \mathcal{S}_{g,2}(\omega_\pi(\sigma_1+\sigma_2))$ of roots of the twisted canonical bundle $\sqrt{\omega_C(p+q)}$ on a marked curve (C, p, q). Similarly, we used $T_{g,2}^2 = \mathcal{S}_{g,2}^2(\omega_\pi(\sigma_1+\sigma_2))$ for the boundary $\Delta_0^{nn} \subset \overline{\mathcal{S}}_g^{--}$.

Let $\overline{T}_{g,2}^2 = \overline{\mathcal{S}}_{g,2}^2(\omega_\pi(\sigma_1+\sigma_2))$ be our compactification of pairs of roots applied to $T_{g,2}^2$. Recall from Section 1.7.3 that $T_{g,2}^2$ and thus $\overline{T}_{g,2}^2$ are irreducible.

We will now define a single type of boundary component of $\overline{T}_{g,2}^2$ which we will use. By $\Delta_i^{++} \subset \overline{T}_{g,2}^2$ we will mean the boundary component whose generic element $(\overline{X}, s_1, s_2, \mathcal{E}_1, \mathcal{E}_2)$ consists of the following data:

- A stable curve $\overline{X} = C_1 \cup_{p\sim q} C_2$ such that $g(C_1) = i$ and two marked points *both of which lie on* C_2, i.e., $s_1, s_2 \in C_2$.
- Two singular roots $\mathcal{E}_1, \mathcal{E}_2$ of $\omega_{\overline{X}}$ where $\mathcal{E}_i$ is obtained from a pair of line bundles (μ_i, τ_i) satisfying $\mu_i \in S_{C_1}^+$ and $\tau_i \in \sqrt{\omega_{C_2}(s_1+s_2)}$. Furthermore, $\mu_1 \not\simeq \mu_2$, $\tau_1 \not\simeq \tau_2$ and $h^0(\mu_i) = 0$, $h^0(\tau_i) = 1$.

Note that $\overline{X}$ is of compact type so that we may ignore the synchronization data.

Define the divisors $D_i = (\mu_i + p, \tau_i)$ which we may assume is supported away from the node by picking $q \in C_2$ generic. Arguing as in Section 2.2 we see that D_i are the limits of base points of roots of the twisted canonical bundle on smooth curves.

Proposition 3.38. *Assume $g \geq 2$. For a general $(C, p, q) \in \mathcal{M}_{g,2}$ and for any distinct pair $\tau_1, \tau_2 \in \sqrt{\omega_C(p+q)}$ we have $\tau_1 \cap \tau_2 = \emptyset$.*

Proof. The moduli space $\overline{T}_{g,2}^2$ is quasi-finite over $\overline{\mathcal{M}}_{g,2}$ and the condition that $(C, p, q, \tau_1, \tau_2) \in \overline{T}_{g,2}^2$ has $\tau_1 \cap \tau_2 = \emptyset$ is open in $\overline{T}_{g,2}^2$ (or more precisely in the locus where $h^0(\tau_i) = 1$). Therefore, it suffices to exhibit one example in $\overline{T}_{g,2}^2$ for which $\tau_1 \cap \tau_2 = \emptyset$ in order to prove the claim.

We will construct this example by degenerating to the boundary $\Delta_1^{++} \subset \overline{T}_{g,2}^2$ and using induction. For the base case, take $g = 2$. Notice that a root $\tau \in \sqrt{\omega_C(p+q)}$ for a smooth genus 2 curve C corresponds to the contact points of a bitangent of the nodal embedding of C into $\mathbb{P}^2 \simeq \mathbb{P}(\mathrm{H}^0(\sqrt{\omega_C(p+q)}))$. Since two distinct bitangents can not have contact points in common, this settles the case $g = 2$.

For $g > 2$ take a degenerate curve $(\overline{X}, s_1, s_2, \mathcal{E}_1, \mathcal{E}_2) \in \Delta_1^{++}$ exactly as described above. We wish to show that $D_1 \cap D_2 = \emptyset$ with D_i defined as above. Since μ_i's are distinct and of degree 0, the degree 1 divisors $\mu_i + p$ are necessarily disjoint. Thus $D_1 \cap D_2 = \emptyset$ iff $\tau_1 \cap \tau_2 = \emptyset$. But $g(C_2) = g - 1$ and $\tau_i \in \sqrt{\omega_{C_2}(s_1 + s_2)}$ so that by induction $\tau_1 \cap \tau_2 = \emptyset$. $\square$

Proposition 3.38 implies that the divisor Δ_0^{nn} does not belong to $\widetilde{\Omega}_g$, which in turn implies that $\overline{\Omega}_g \cap \Delta_0^{nn} = \widetilde{\Omega}_g \cap \Delta_0^{nn}$ in $\overline{\mathcal{U}}_g^{--}$.

We wish to show that the content of Proposition 3.38 holds if the points $p, q \in C$ are taken to be a general theta marking. This would then ensure that the $f_{\overline{X}}^{nn}$ defined in Section 3.1 will equal 0. We can not do this however, so we will state it as a conjecture and specialize to a problem which is easier to state and will nevertheless imply this conjecture.

Conjecture 3.39. *If (C, p, q) is a general curve with general theta markings then for every distinct pair of roots $\tau_1, \tau_2 \in \sqrt{\omega_C(p + q)}$ we have $\tau_1 \cap \tau_2 = \emptyset$.*

We proved in Theorem III.1.8 that the moduli space of curves with theta markings Z_2 is irreducible. Thus, it is sufficient to exhibit one example $(C, \eta_1, \eta_2, D, p, q) \in Z_2$ such that for all distinct $\tau_1, \tau_2 \in \sqrt{\omega_C(p + q)}$ we have $\tau_1 \cap \tau_2 = \emptyset$. Clearly, for any hyperelliptic curve C and pair of Weierstrass points $w_1, w_2 \in C$ we can realize (w_1, w_2) as a (possibly degenerate) theta marking associated to a pair of odd theta characteristics. This implies:

Lemma 3.40. *If there exists a hyperelliptic curve C and a pair of Weierstrass points $w_1, w_2 \in C$ such that for every pair of distinct roots $\tau_1, \tau_2 \in \sqrt{\omega_C(w_1 + w_2)}$ we have $\tau_1 \cap \tau_2 = \emptyset$ then Conjecture 3.39 holds.*

3.6 Reducible double spin curves

Recall from Example 1.15 that if (X, L) is a general quasi-stable spin curve in $\Delta_i^+ \subset \overline{\mathcal{S}}_g^-$ then $X = C_1 \cup_{p\sim 0} \mathbb{P}^1 \cup_{\infty\sim q} C_2$ for $g(C_1) = i$, $\eta_1 := L|_{C_1}$ is an even theta characteristic on C_1 and $\eta_2 := L|_{C_2}$ is an odd theta characteristic on C_2.

Moreover, we showed in Section 2.2 that if (X, L) is contact general in the sense of Definition 2.13 then the limit contact divisor D of X satisfies the following:

$$D = \begin{cases} \eta_1 + p & : \text{ on } C_1 \\ \emptyset & : \text{ on } \mathbb{P}^1 \\ \eta_2 & : \text{ on } C_2 \end{cases}$$

Therefore, we may as well work with the stable model of (X, L) since the exceptional component $\mathbb{P}^1$ plays no role here.

For the following subsections, we fix $(X, L_1, L_2) \in \Delta_i^{xy} \subset \overline{\mathcal{S}}_g^{--}$ where each (X, L_i) is assumed to be theta general. Let the stable model of (X, L_1, L_2) be $(\overline{X}, \mathcal{E}_1, \mathcal{E}_2)$ which we will often just denote by $(\overline{X}, \Xi)$ where Ξ is the matrix $(\eta_{ij})_{i,j=1}^2$ with entries $\eta_{ij} := L_i|_{C_j}$. For each $i = 1, 2$ let $D_i \subset \overline{X}$ be the contact divisor of $\mathcal{E}_i$.

3.6.1 On Δ_i^{++}

The contact divisors are of the following form:

$$D_1 = \begin{cases} \eta_{11} + p & : \text{ on } C_1 \\ \eta_{12} & : \text{ on } C_2 \end{cases} \quad \text{and} \quad D_2 = \begin{cases} \eta_{21} + p & : \text{ on } C_1 \\ \eta_{22} & : \text{ on } C_2 \end{cases}$$

Therefore, $D_1 \cap D_2 \neq \emptyset$ iff one of the following two conditions are satisfied:

- Either $(\eta_{11} + p) \cap (\eta_{21} + p) \neq \emptyset$,
- or $\eta_{12} \cap \eta_{22} \neq \emptyset$.

The second condition will hold iff $(C_2, \eta_{12}, \eta_{22}) \in \Omega_{g-i}$ and the marked point $q \in C_2$ plays no role. This is a divisorial condition in Δ_i^{++}. The first condition will hold if $p \in C_1$ is one of finitely many Scorza points associated to (η_{11}, η_{21}). For C_1 general, there will be exactly $2i(i-1)$ such points on C_1, see Section 3.7.

In particular, if $(\overline{X}, \Xi)$ is general we have $D_1 \cap D_2 = \emptyset$ which implies $\overline{\Omega}_g \cap \Delta_i^{++} = \widetilde{\Omega}_g \cap \Delta_i^{++}$ in $\overline{\mathcal{U}}_g^{--}$.

3.6.2 On Δ_i^{+-}

This time the contact divisors are of the following form:

$$D_1 = \begin{cases} \eta_{11} + p & : \text{ on } C_1 \\ \eta_{12} & : \text{ on } C_2 \end{cases} \quad \text{and} \quad D_2 = \begin{cases} \eta_{21} & : \text{ on } C_1 \\ \eta_{22} + q & : \text{ on } C_2 \end{cases}$$

Therefore, $D_1 \cap D_2 \neq \emptyset$ iff one of the following two conditions are satisfied:

- Either $(\eta_{11} + p) \cap \eta_{21} \neq \emptyset$,
- or $\eta_{12} \cap (\eta_{22} + q) \neq \emptyset$.

Given $(C, \mu, \eta) \in \mathcal{S}_g^{+-}$ which is rigid let us make the following definition:

Definition 3.41. A point $p \in C$ is called a *switch point of* (μ, η) if $(\mu + p) \cap \eta \neq \emptyset$.

The name comes from the fact that a proper analysis of such points comes by switching the role of p with a contact point in the following sense. For any $p \in C$ we have $q \in (\mu + p) \cap \eta$ iff $p \leq \mu + q$. In other words, if we write $\eta = q_1 + \cdots + q_{g-1}$ then the collection of switch points of (μ, η) can be expressed as the union

$$\bigcup_{i=1}^{g-1} (\mu + q_i).$$

In particular, this union is finite. This implies that the general element $(\overline{X}, \Xi) \in \Delta_i^{+-}$ will have $D_1 \cap D_2 = \emptyset$. Thus $\overline{\Omega}_g \cap \Delta_i^{+-} = \widetilde{\Omega}_g \cap \Delta_i^{+-}$ in $\overline{\mathcal{U}}_g^{--}$.

Remark 3.42. In a treatment analogous to the Scorza points given in Section 3.7, one can define the locus of switch points in $\mathcal{S}_{g,1}^{+-}$ and proceed to show that it is irreducible. This implies that for a general curve C and any switch point p of (μ, η) we have $\#(\mu+p) \cap \eta = 1$. We omit the proofs because we will not use this result but also because the line of reasoning is almost identical to that in Section 3.7.

3.7 Scorza points

This section is not strictly necessary for the main results. However, this topic is fundamental to a more comprehensive study of the boundary $\overline{\Omega}_g \cap \Delta_i^{++}$ of $\overline{\Omega}_g$. Therefore, we set the ground for further analysis. In particular, Corollary 3.51 maybe of independent interest.

Let $(C, p, \mu) \in \mathcal{S}_{g,1}^+$ with $h^0(\mu) = 0$. Then we have $h^0(\mu + p) = 1$. In general, the divisor $\mu + p$ is reduced: since $\mathcal{S}_{g,1}^+$ is irreducible, it suffices to check this on one example and the hyperelliptic curves provide many.

Definition 3.43. Let $(C, \mu_1, \mu_2) \in \mathcal{S}_g^{++}$ be rigid. Then any point $p \in C$ such that $(\mu_1 + p) \cap (\mu_2 + p) \neq \emptyset$ will be called a *Scorza point.*

Definition 3.44. For rigid $(C, \mu) \in \mathcal{S}_g^+$ the locus $\Gamma_\mu = \{(p, q) \mid q \leq \mu + p\} \subset C \times C$ is one dimensional and is called the *Scorza correspondence* (§5.5 of [Dol12]).

Remark 3.45. Note that Γ_μ is symmetric, i.e. $(p, q) \in \Gamma_\mu$ iff $(q, p) \in \Gamma_\mu$. Furthermore, the projection maps realize Γ_μ as a degree g cover of C. It has been recently proven that Γ_μ is a smooth irreducible curve when C is generic ([TZ11],[FV14]).

Let $(C, \mu_1, \mu_2) \in \mathcal{S}_g^+$ be rigid, $\Gamma_i \subset C \times C$ be the Scorza correspondence of μ_i and $\Gamma_{12} := \Gamma_1 \cap \Gamma_2$ be the intersection. Since Γ_i's are irreducible in general, Γ_{12} is supported on finitely many points. Furthermore, Γ_{12} is always symmetric so that for the two projections $\mathrm{pr}_i : C^2 \to C$ the image set $\mathrm{pr}_i(\Gamma_{12})$ is independent of the projection used.

Remark 3.46. The image $\mathrm{pr}_i(\Gamma_{12}) \subset C$ is supported on the Scorza points of C.

Proposition 3.47. *For a general $(C, \mu_1, \mu_2) \in \mathcal{S}_g^{++}$ there are precisely $2g(g-1)$ Scorza points.*

Proof. It is easy to show (for example, see §5.5 of [Dol12]) that the divisor classes $[\Gamma_i] \in \mathrm{Pic}(C^2)$ satisfy the following equality:

$$[\Gamma_i] \equiv \mathrm{pr}_1^*[\mu_i] + [\Delta] + \mathrm{pr}_2^*[\mu_i].$$

Therefore $[\Gamma_1] \cdot [\Gamma_2] = 2g(g-1)$.

We prove in Corollary 3.51 that, for general (C, μ_1, μ_2), Γ_1 and Γ_2 intersect transversally. Furthermore, in Corollary 3.52 we prove that, for general (C, μ_1, μ_2) and any Scorza point p associated to (μ_1, μ_2) we have $\#(\mu_1 + p) \cap (\mu_2 + p) = 1$. In other words, the projections pr_i map each point of Γ_{12} to a different point on C. Putting all this together, we get the desired result. $\square$

Let $\mathcal{U}_g^{++} \subset \mathcal{S}_g^{++}$ be a non-empty open locus such that the theta characteristics are rigid and the number of intersection points, $\Gamma_1 \cap \Gamma_2$, of the corresponding Scorza correspondences is finite. Denoting the universal curve over $\mathcal{U}_g^{++}$ by $\mathcal{C} \to \mathcal{U}_g^{++}$ one can easily construct the *relative Scorza correspondences* $\mathfrak{G}_1, \mathfrak{G}_2 \subset \mathcal{C} \times_{\mathcal{U}_g^{++}} \mathcal{C}$ and define the intersection $\mathfrak{G}_{12} = \mathfrak{G}_1 \cap \mathfrak{G}_2$.

Notation 3.48. Let $\Omega_g^+ \subset \mathcal{C} \subset \mathcal{S}_{g,1}^{++}$ be the image of $\mathfrak{G}_{12}$ via any of the projections. We will call Ω_g^+ the *locus of Scorza points.*

Proposition 3.49. *The locus $\mathfrak{G}_{12}$ is irreducible.*

Proof. Each fiber of $\mathfrak{G}_{12} \to \mathcal{U}_g^{++}$ is quasi-finite of degree $2g(g-1)$, counting multiplicities, as is evident from the proof of Proposition 3.47. Since $\mathfrak{G}_{12}$ is closed in $\mathcal{C}^2$ and $\mathcal{C}^2 \to \mathcal{U}_g^{++}$ is proper, the map $\mathfrak{G}_{12} \to \mathcal{U}_g^{++}$ is finite. Thus, it remains to show that the monodromy action of $\mathfrak{G}_{12} \to \mathcal{U}_g^{++}$ on a single fiber is transitive.

We do this by degenerating to a hyperelliptic curve and a particular pair of even theta characteristics, where it is clear that the monodromy is transitive on the intersection points.

Let C be a hyperelliptic curve with marked Weierstrass points $W = \{w_1, \ldots, w_{2g+2}\}$. For any $I \subset [2g+2]$ of size $g+1$, we define the theta characteristic $\eta_I := \sum_{i \in I} w_i - \mathfrak{g}_2^1$. Then:

$$\eta_I + w_i = \begin{cases} w_{I \setminus \{i\}} & : \ i \in I \\ w_{I^c \setminus \{i\}} & : \ i \notin I \end{cases}$$

Let $I = [g+1]$ and $J = [g+1, 2g+1]$. It will be clear that $\xi := (C, \eta_I, \eta_J)$ belongs to $\mathcal{U}_g^{++}$ and we will show that the monodromy action of $\mathfrak{G}_{12} \to \mathcal{U}_g^{++}$ over ξ is transitive.

Indeed, $\mathfrak{G}_{12}|_\xi$ is identified with the set of points $\{(w_i, w_j) \in C^2 \mid i \neq j, i, j \in I \cap J^c \text{ or } i, j \in I^c \cap J\}$. Note here that there are precisely $2g(g-1)$ points in this set, which is the degree of $\mathfrak{G}_{12}$ over $\mathcal{U}_g^{++}$ so none of these points are ramification points of $\mathfrak{G}_{12} \to \mathcal{U}_g^{++}$.

We now describe the monodromy action on the Weierstrass points fixing ξ. The monodromy can permute elements in $I \cap J^c$ as well as permute elements in $I^c \cap J$ without restriction. Furthermore, we can fix η_I by mapping I to I^c and similarly with η_J. It is straightforward to check that using this symmetry we can map $I \cap J^c$ to $I^c \cap J$ via any bijective map. Hence the monodromy action is transitive on $\{(w_i, w_j) \in C^2 \mid i \neq j, i, j \in I \cap J^c \text{ or } i, j \in I^c \cap J\}$. $\square$

Corollary 3.50. *The locus of Scorza points Ω_g^+ is irreducible.*

Proof. The locus Ω_g^+ is by definition the image of $\mathfrak{G}_{12}$, which is irreducible. $\square$

Corollary 3.51. *For general $(C, \mu_1, \mu_2) \in \mathcal{S}_g^{++}$ the Scorza correspondences Γ_1 and Γ_2 intersect transversally in C^2.*

Proof. Non-transversal intersections correspond to ramification points of $\mathfrak{G}_{12} \to \mathcal{U}_g^{++}$. The ramification locus in $\mathfrak{G}_{12}$ is closed and $\mathfrak{G}_{12}$ is irreducible. Therefore, the existence of one point in $\mathcal{U}_g^{++}$ over which the map $\mathfrak{G}_{12} \to \mathcal{U}_g^{++}$ is unramified will be sufficient to prove the claim.

The intersection Γ_{12} constructed over the hyperelliptic curve appearing in the proof of Proposition 3.49 consists of precisely $2g(g-1)$ distinct points. Since that is the degree of the intersection product $[\Gamma_1] \cdot [\Gamma_2]$, Γ_1 and Γ_2 intersect transversally along Γ_{12}. $\square$

Corollary 3.52. *If $(C, p, \mu_1, \mu_2) \in \Omega_g^+$ is general then $\#(\mu_1 + p) \cap (\mu_2 + p) = 1$.*

Proof. Since Ω_g^+ is irreducible, it suffices to check this property on a single element and apply semi-continuity theorem. Let C be a hyperelliptic curve with Weierstrass points $\{w_1, \ldots, w_{2g+2}\}$ and consider $I = [g+1]$ and $J = [g, 2g]$. Then $\eta_I + w_{g+1} \equiv \sum_{i=1}^{g} w_i$ and $\eta_J + w_{g+1} \equiv w_g + \sum_{i=g+2}^{2g} w_i$. Therefore $\#(\eta_I + w_{g+1}) \cap (\eta_J + w_{g+1}) = 1$. $\square$

Chapter 4

Components of the moduli space

As we mentioned in Section I.1.2, the compactified moduli spaces of m-tuples of roots, $\overline{\mathcal{S}}^m(\mathcal{N})$, can be defined over $\mathbb{Z}[\frac{1}{2}]$ as opposed to a field. Moreover, we proved that $\overline{\mathcal{S}}^m(\mathcal{N})$ is proper and smooth over $\mathbb{Z}[\frac{1}{2}]$ so that we can apply Theorem 4.17.(iii) of [DM69] to conclude that for any algebraically closed field k of char $\neq 2$, the number of connected components of $\overline{\mathcal{S}}^m(\mathcal{N})|_k$ is the same as the number of components of $\overline{\mathcal{S}}^m(\mathcal{N})|_{\mathbb{C}}$. Since, these spaces are smooth, their connected components and irreducible components coincide.

In particular, the boundary $\overline{\mathcal{S}}^m(\mathcal{N}) \setminus \mathcal{S}^m(\mathcal{N})$ can be ignored and we may simply work with m-tuples of roots over a smooth curve. To that end, let $\mathcal{S}_g^{\times m} = \mathcal{S}_g \times_{\mathcal{M}_g} \cdots \times_{\mathcal{M}_g} \mathcal{S}_g$ where we take an m-fold fiber product. Our goal in this section is to count the number of components of $\mathcal{S}_g^{\times m}$.

As we discussed, regardless of the original base field, our question can simply be answered by studying the number of components of the complex points of $\mathcal{S}_g^{\times m}$. Viewing $\mathcal{S}_g^{\times m}$ as a stack, the groupoid of complex points naturally has the structure of a complex orbifold, whose components we will count.

4.1 Teichmüller spaces, spin structures and quadratic forms

Let $\Gamma = \Gamma_g$ be the mapping class group of a real surface F of genus g. We will denote by $\mathcal{T} = \mathcal{T}_g$ the Teichmüller space of F. See [EF85] and [Gro62] for references on this subject. We will let $V = \mathrm{H}^1(F, \mathbb{F}_2)$, which is to be viewed as a $2g$-dimensional $\mathbb{F}_2$-vector space.

Let X be a Riemann surface of genus g and let $S(X)$ denote the set of spin structures on X up to isomorphism. Let $\mathrm{Pic}_X[2]$ be the 2-torsion subgroup of Pic_X. Any homeomorphism $f : F \xrightarrow{\sim} X$ induces an isomorphism $f^* : \mathrm{Pic}_X[2] \xrightarrow{\sim} V$ by viewing Pic_X as the quotient $\mathrm{H}^1(X, \mathcal{O}_X)/\mathrm{H}^1(X, \mathbb{Z})$ and V as the quotient $\mathrm{H}^1(X, \frac{1}{2}\mathbb{Z})/\mathrm{H}^1(X, \mathbb{Z})$.

The mapping class group acts on $\mathrm{H}^1(F, \mathbb{Z})$ in a way that preserves the intersection product. This action induces a *surjective* map $\Gamma_g \twoheadrightarrow \mathrm{Sp}(V)$, where we omit the intersection product from notation. Given a Teichmüller marking f on X, the mapping class group acts on $\mathrm{Pic}_X[2]$ as the full spin group preserving the Weil pairing.

Let Q be the affine space of quadratic forms on V. Having fixed $f : F \xrightarrow{\sim} X$, the set $S(X)$ admits a canonical identification with Q in the following way. Given $\eta \in S(X)$ the map

$$\begin{aligned} q_\eta : V &\to \mathbb{F}_2 \\ \gamma &\mapsto h^0(\eta + (f^*)^{-1}(\gamma)) + h^0(\eta) \pmod 2 \end{aligned}$$

is a non-singular quadratic form on V with $\mathrm{Arf}(q_\eta) \equiv h^0(\eta) \pmod 2$, see [Mum71] and see the next section for more on Arf invariants. The function $S(X) \to Q : \eta \mapsto q_\eta$ is a bijection and Γ acts on Q by precomposition: $Q \times \Gamma \to Q : (q, \gamma) \mapsto q \circ \gamma$.

It is well known that the orbifold quotient $\lfloor \mathcal{T}/\Gamma \rfloor$ is a fine moduli space for Riemann surfaces in the differential category. Due to the modular interpretation of both spaces, the complex points of the *fine moduli space* $\mathcal{M}_g$ (with the groupoid structure) are isomorphic to the orbifold quotient $\lfloor \mathcal{T}_g/\Gamma \rfloor$.

When working with these fine moduli spaces, we need not worry about the fact that non-trivial automorphisms of some curves can identify non-isomorphic theta characteristics. In particular, the morphism of fine moduli spaces $\mathcal{S}_g \to \mathcal{M}_g$ is unramified. Since $\mathcal{T}_g$ is simply connected, the pullback of $\mathcal{S}_g$ to $\mathcal{T}_g$ will then be the disjoint union of 2^{2g} copies of $\mathcal{T}_g$.

The monodromy of $\mathcal{S}_g \to \mathcal{M}_g$ over X is then precisely the permutation action of the mapping class group Γ on the sheets of the cover $\rho : \mathcal{S}_g|_{\mathcal{T}_g} \to \mathcal{T}_g$. For each $[X, g] \in \mathcal{T}_g$, the equivalence class of X with a Teichmüller marking g, the fiber $\rho^{-1}([X, g])$ can naturally be identified with the spin structures $S(X)$ on X. The following statement appears to be well known, e.g., see [Sip82], and in any case is not hard to prove.

Lemma 4.1. *The monodromy action of Γ_g on the fiber $S(X)$, after the identification $S(X) = Q$, coincides with the pre-composition action of Γ_g on Q.*

Therefore, the monodromy of the finite cover $\mathcal{S}_g \to \mathcal{M}_g$ is equivalent to the action of $\mathrm{Sp}(V)$ on Q by precomposition. Using the m-fold product, and the irreducibility of $\mathcal{M}_g$, this gives the following result:

Lemma 4.2. *The number of components of $\mathcal{S}_g^{\times m}$ is in bijection with the $\mathrm{Sp}(V)$-orbits of $Q^{\times m} = Q \times \cdots \times Q$ acting by pre-composition: for $\gamma \in \mathrm{Sp}(V)$ this action is $(q_1, \ldots, q_m) \cdot \gamma = (q_1 \circ \gamma, \ldots, q_m \circ \gamma)$.*

4.2 Affine geometry of quadratic forms

The standard text book [Art88] offers a comprehensive treatment of quadratic spaces but in char $\neq 2$. We will work with these spaces in characteristic 2 but the basic treatment is the same. See [GH04] for precisely our point of view and for a survey of the connection between quadratic forms, theta characteristics and Arf invariants.

Let V be a $2g$-dimensional vector space over $\mathbb{F}_2$. A bilinear form $f : V \times V \to \mathbb{F}_2$ is called *symplectic* if $f(v, v) = 0$ for all $v \in V$. The tuple (V, f), or f, is called *non-singular* if $V \to V^\vee : v \mapsto f(v, \cdot)$ is an isomorphism. A map $q : V \to \mathbb{F}_2$ is called a quadratic form on V associated to f_q if:

- $q(0) = 0$,
- $f_q : V \times V \to \mathbb{F}_2$ defined by $f_q(v, w) = q(v + w) + q(v) + q(w)$ is bilinear.

A tuple (V, q) where q is a quadratic form is called a *quadratic space* and it is called non-singular if the associated bilinear form f_q is non-singular. A morphism $(V, q) \to (V', q')$ between quadratic spaces is a linear map $\varphi : V \to V'$ satisfying $q = q' \circ \varphi$.

Let V be a $2g$-dimensional vector space over $\mathbb{F}_2$ with a non-singular symplectic inner product $f : V \times V \to \mathbb{F}_2$. Let Q be the set of quadratic forms on V associated to the inner product f. The set Q is an affine space with space of translations V. The translation action is given by $V \times Q \to V : (v, q) \mapsto q + f(v, \cdot)$. We will simply write the latter quadratic form as $q + v$. Moreover, if $q, q' \in Q$ are such that $q' = q + v$ then we will express v as $q' - q$, or $q' + q$ as the characteristic is 2.

Let $\mathrm{Sp}(V)$ be the group of linear automorphisms of V respecting the symplectic inner product f. There is a natural action of $\mathrm{Sp}(V)$ on Q via pre-composition. In other words, there is a pairing $\mathrm{Sp}(V) \times Q \to Q : (\varphi, q) \mapsto \varphi_* q = q \circ \varphi^{-1}$. Notice that $\varphi_*(q + v) = \varphi_* q + \varphi(v)$ since f is invariant under φ.

Remark 4.3. The identity $\varphi_*(q+v) = \varphi_* q + \varphi(v)$ is why we use the left action of $\mathrm{Sp}(V)$ instead of the right action. Note that this does not change the orbits of the action.

Let $S = (q_1, \ldots, q_m)$ and $S' = (q'_1, \ldots, q'_m)$ be sequences of quadratic forms on V. If $\exists \varphi \in \mathrm{Sp}(V)$ such that $\varphi_* S = S'$ then let us write $S \sim S'$ and call S and S' equivalent. We would like to know under what circumstances S is equivalent to S'.

If there is a relation in S of the form $q_k = q_1 + q_i + q_j$ then $S \sim S'$ can only hold if the corresponding relation $q'_k = q'_1 + q'_i + q'_j$ holds in S'. For this reason, we may as well delete the k-th quadratic from both sides and reduce the problem to that of fewer quadratics. This prompts the following definition.

Definition 4.4. A sequence of quadratic forms $S = (q_1, \ldots, q_m)$ is called *non-degenerate* if the affine span of S in Q is of dimension $m - 1$ and called *degenerate* otherwise.

Remark 4.5. For a sequence of theta characteristics $(\eta_1, \ldots, \eta_m)$ on a smooth curve C, the corresponding sequence of quadratic forms on $\mathrm{H}^1(C, \mathbb{Z})$ is degenerate iff there exists a subsequence $(\eta_{i_1}, \ldots, \eta_{i_{2l}})$ of even length such that

$$\eta_{i_1} \otimes \cdots \otimes \eta_{i_{2l}} \simeq \omega_C^{\otimes l}.$$

We will now assume S and S' are non-degenerate. For each $q \in Q$ let $\mathrm{Arf}(q) \in \mathbb{F}_2$ denote the Arf invariant of q. If $m = 1$ then it is well known that $S \sim S'$ iff $\mathrm{Arf}(q_1) = \mathrm{Arf}(q'_1)$. If $m = 2$ then Proposition 4.9 below implies that $S \sim S'$ iff $\mathrm{Arf}(q_1) = \mathrm{Arf}(q'_1)$ and $\mathrm{Arf}(q_2) = \mathrm{Arf}(q'_2)$. For $m \geq 2$ more invariants begin to appear. In order to deal with the added complexity, we will use Witt's Lemma.

Lemma 4.6 (Witt's Lemma)**.** *Let (V, q) and (V', q') be non-singular quadratic spaces which are isometric. Let $W \subset V$ and $W' \subset V'$ be subspaces and $\mu : W \xrightarrow{\sim} W'$ be an isometry with respect to the induced quadratic forms $q|_W$ and $q|_{W'}$. Then there is an isometry $\varphi : V \to V'$ such that $\varphi|_W = \mu$.*

Proof. See Theorem 3.3 and Exercise 3.31 in [Wil09] for a proof. □

Notation 4.7. Given S we define a sequence $\mathfrak{a} = (a_i, a_{kl})$ as follows. Let $a_i = \mathrm{Arf}(q_i)$ for $i = 1, \dots, m$. Define quadratic forms $q_{kl} = q_1 + q_k + q_l$ for each $k \neq l$. Set $a_{kl} = \mathrm{Arf}(q_{kl})$.

Notation 4.8. Given S we define a sequence $\mathfrak{e} = (e_i, e_{kl})_{i,j}$ as follows. For each $i = 2, \dots, m$ let $v_i = q_i + q_1$, $e_i = q_1(v_i)$ and for each $k \neq l$ let $e_{kl} = f(v_k, v_l)$. Let $W = \langle v_2, \dots, v_m \rangle \subset V$.

By definition, S is non-degenerate iff $\dim W = m - 1$. Note also that we have $\mathrm{Arf}(q_i) = \mathrm{Arf}(q_1 + v_i) = \mathrm{Arf}(q_1) + q_1(v_i)$ so $e_i = a_1 + a_i$. Similarly one can check $e_{ij} = a_1 + a_i + a_j + a_{ij}$. Therefore, we can obtain $\mathfrak{e}$ from $\mathfrak{a}$. Provided that we know a_1 we can go in the other direction and recover $\mathfrak{a}$ from $\mathfrak{e}$.

For $S' = (q'_1, \dots, q'_m)$, let $\mathfrak{a}' = (a'_i, a'_{ij})$, $\mathfrak{e}' = (e'_i, e'_{ij})$ and $W' = \langle v'_2, \dots, v'_m \rangle$ as with S.

Proposition 4.9. *Let $S = (q_1, \dots, q_m)$ and $S' = (q'_1, \dots, q'_m)$ be non-degenerate sequences of quadratics on a non-singular symplectic space (V, f). Define $\mathfrak{a}$ and $\mathfrak{a}'$ as above. We can find $\varphi \in \mathrm{Sp}(V)$ such that $\varphi_* S = S'$ iff $\mathfrak{a} = \mathfrak{a}'$.*

Proof. Elements of $\mathrm{Sp}(V)$ will preserve the Arf invariant. Therefore, if $\varphi_* S = S'$ then the associated sequence of Arf invariants are equal, i.e., $\mathfrak{a} = \mathfrak{a}'$.

Assume now $\mathfrak{a} = \mathfrak{a}'$. Then, $\mathrm{Arf}(q_1) = \mathrm{Arf}(q'_1)$ and $\mathfrak{e} = \mathfrak{e}'$. In particular, the (non-singular) quadratic spaces (V, q_1) and (V, q'_1) are isometric.

Let $W = \langle v_2, \dots, v_m \rangle$ and $W' = \langle v'_2, \dots, v'_m \rangle$ be defined as above. Since we assumed S and S' are non-degenerate the vectors $\{v_i\}$ and $\{v'_i\}$ form a basis for W and W' respectively. Therefore we can define a linear map $\mu : W \to W' : v_i \mapsto v'_i$.

Note that $q_1(v_i) = e_i = e'_i = q'_1(v'_i)$ and $f(v_i, v_j) = e_{ij} = e'_{ij} = f(v'_i, v'_j)$. This implies that $\mu : (W, q_1|_W) \to (W', q'_1|_{W'})$ is in fact an isometry. Applying Witt's Lemma we conclude that there is an isometry $\varphi : (V, q_1) \to (V, q'_1)$ extending μ.

Necessarily $\varphi \in \mathrm{Sp}(V)$ and $\varphi_* q_1 = q'_1$. Furthermore, $\varphi_*(q_i) = \varphi_*(q_1 + v_i) = \varphi_*(q_1) + \varphi(v_i) = q'_i$. Thus $\varphi_* S = S'$. □

4.2.1 Existence of sequences

Let $N = m + \binom{m-1}{2}$. The sequence $\mathfrak{a} = (a_i, a_{kl})$ of Notation 4.7 can be viewed as an element of $\mathbb{F}_2^N$ since $a_{kl} = a_{lk}$. We now ask, given $\mathfrak{a} \in \mathbb{F}_2^N$ can we find a non-degenerate sequence of quadratic forms $(q_1, \dots, q_m)$ such that $\mathrm{Arf}(q_i) = a_i$ and $\mathrm{Arf}(q_1 + q_k + q_l) = a_{kl}$.

Let $\mathfrak{e} = (e_i, e_{kl})$ be defined by $e_i = a_i + a_1$ for $i = 2, \dots, m$ and $e_{kl} = a_{kl} + a_k + a_l + a_1$. Let $W = \langle v_2, \dots, v_m \rangle$ be an abstract space with (possibly degenerate) symplectic form $v_i \cdot v_j = e_{ij}$ and an associated quadratic form $q_W(v_i) = e_i$.

The discussion in the previous section implies that the existence of S corresponding to $\mathfrak{a}$ is equivalent to the existence of an isometric immersion $(W, q_W) \hookrightarrow (V, q_V)$ where q_V is any quadratic form on V with $\mathrm{Arf}(q_V) = a_1$. We will now solve this existence problem completely.

Let E be the symmetric off-diagonal matrix corresponding to the symplectic pairing on W. That is $E = (e_{ij})_{ij}$ with $e_{ii} := 0$. We will also view E as the morphism

$W \to W^\vee : w \mapsto (w \cdot \_)$. Set $K = \ker E \subset W$ and pick any $U \subset W$ such that $W = K \oplus U$. Note that U is non-degenerate and the value of $\mathrm{Arf}(q_W|_U)$ does not depend on our choice of U.

Remark 4.10. For the following, use the convention that the quadratic form on the zero space is zero and has Arf invariant zero.

Definition 4.11. Let $c \in \{0, 1\}$ be the *correction term* defined as follows:

$$c = \begin{cases} 0 & : \quad q_W|_K \not\equiv 0 \\ 0 & : \quad q_W|_K \equiv 0,\ \mathrm{Arf}(q_W|_U) = \mathrm{Arf}(q_V) \\ 1 & : \quad q_W|_K \equiv 0,\ \mathrm{Arf}(q_W|_U) \neq \mathrm{Arf}(q_V) \end{cases}$$

Proposition 4.12. *There is an isometric immersion $(W, q_W) \hookrightarrow (V, q_V)$ iff the following inequality is satisfied:*

$$\dim V \geq \dim W + \dim K + 2c.$$

Proof. Recall our decomposition $W = K \oplus U$. Let K' be an isomorphic copy of K and define $W' = W \oplus K'$. Extend the symplectic pairing on W to W' so that K' is orthogonal to U and is dual to K. Naturally, W' is a non-singular symplectic space and any embedding $W \hookrightarrow V$ will extend to an embedding $W' \hookrightarrow V$. This forces the inequality $\dim V \geq \dim W + \dim K$.

Pick a basis $\gamma_1, \dots, \gamma_s$ of K and its dual basis $\gamma'_1, \dots, \gamma'_s \in K'$. Any extension of the quadratic form q_W to a quadratic form $q_{W'}$ on W' requires only the values $q_{W'}(\gamma'_i)$ for $i = 1, \dots, s$. Observe that:

$$\mathrm{Arf}(q_{W'}) = \mathrm{Arf}(q_W|_U) + \sum_{i=1}^{s} q_W(\gamma_i) q_{W'}(\gamma'_i).$$

If $q_W|_K \not\equiv 0$ then we can choose an extension $q_{W'}$ of either Arf invariant. However, if $q_W|_K \equiv 0$ then any extension $q_{W'}$ will necessarily have $\mathrm{Arf}(q_{W'}) = \mathrm{Arf}(q_W|_U)$. To change the parity, we would have to join an odd plane to W', increasing the dimension by 2 and obtaining, say $(W'', q_{W''})$. This forces the refined inequality $\dim V \geq \dim W + \dim K + 2c$.

However, this inequality is also sufficient. Simply construct $(W', q_{W'})$ (or the larger $(W'', q_{W''})$ if necessary) such that $\mathrm{Arf}(q_{W'}) = \mathrm{Arf}(q_V)$ (or $\mathrm{Arf}(q_{W''}) = \mathrm{Arf}(q_V)$). Using the hyperbolic decompositions of V and W' (or W'') it is clear that we can find an embedding $W' \hookrightarrow V$ (or $W'' \hookrightarrow V$). □

The following is an easier inequality, but gives only one direction.

Corollary 4.13. *An isometric immersion $W \hookrightarrow V$ exists if*

$$g = \frac{\dim V}{2} \geq \dim W + 1.$$

Proof. Simply observe that $\dim W \geq \dim K$ and $1 \geq c$ so that the result above applies. □

4.3 Classification of components

Let $\pi : \mathcal{C} \to B$ be a family of smooth curves and $\mathcal{L}_1, \mathcal{L}_2$ be two line bundles on $\mathcal{C}$. As is customary, we will write $\mathcal{L}_1 \sim \mathcal{L}_2$ if there exists a line bundle $\mathcal{N}$ on B such that $\mathcal{L}_1 \simeq \mathcal{L}_2 \otimes \pi^*\mathcal{N}$. Note that in this case $\mathcal{N} = \pi_*(\mathcal{L}_1 \otimes \mathcal{L}_2^\vee)$.

Remark 4.14. If $\mathcal{L}_1$ and $\mathcal{L}_2$ are spin structures on $\mathcal{C} \to B$ then $\mathcal{L}_i^{\otimes 2} \simeq \omega_\pi$. Thus $(\mathcal{L}_1 \otimes \mathcal{L}_2^\vee)^{\otimes 2} \simeq \mathcal{O}_\mathcal{C}$. If $\mathcal{L}_1 \sim \mathcal{L}_2$ and $\mathcal{N} = (\mathcal{L}_1 \otimes \mathcal{L}_2^\vee)$ then $\pi^*\mathcal{N}^{\otimes 2} \simeq \mathcal{O}_\mathcal{C}$. By the projection formula $\mathcal{N}^{\otimes 2} \simeq \pi_*\mathcal{O}_\mathcal{C} \simeq \mathcal{O}_B$, where the latter isomorphism is standard for, e.g., families of curves. In other words, two spin structures are equivalent iff they differ by a 2-torsion pulled back from the base.

Suppose that $\mathcal{C} \to B$ is a family of spin curves and $\mathcal{L}_1, \ldots, \mathcal{L}_m$ are spin structures. Some of these $\mathcal{L}_i$ are redundant if, for a subsequence $\mathcal{L}_{i_1}, \ldots, \mathcal{L}_{i_{2l}}$ of even length, the following relation holds (see Remark 4.5):

$$(\mathcal{L}_{i_1} \otimes \cdots \otimes \mathcal{L}_{i_{2l}}) \sim \omega_{\mathcal{C}/B}^{\otimes l}. \tag{4.3.1}$$

Definition 4.15. A connected component of $\mathcal{S}_g^{\times m}$ where one of the relations 4.3.1 hold will be called a *degenerate component*. The rest of the components will be called *non-degenerate.*

Remark 4.16. If a component $T \subset \mathcal{S}_g^{\times m}$ is degenerate, then some of the m spin structures over T can be expressed in terms of the others and pullbacks of 2-torsion line bundles from T. Thus T is isomorphic to a non-degenerate component of $\mathcal{S}_g^{\times n}$ for some $n < m$. Therefore, it is sufficient to study non-degenerate components of $\mathcal{S}_g^{\times m}$ for all $m \geq 1$.

Given $m \geq 1$ take $\mathfrak{a} \in \mathbb{F}_2^{m+\binom{m-1}{2}}$, where we write $\mathfrak{a} = (a_i; a_{kl})$ for $i \in \{1, \ldots, m\}$ and $k < l \in \{2, \ldots, m\}$.

Definition 4.17. Let $\mathcal{S}_g^\mathfrak{a} \subset \mathcal{S}_g^{\times m}$ be the (possibly empty) non-degenerate component of $\mathcal{S}_g^{\times m}$ on which the following are satisfied:

$$h^0(\eta_i) \equiv a_i \pmod 2$$
$$h^0(\eta_i \otimes \eta_j \otimes \eta_1^{-1}) \equiv a_{ij} \pmod 2$$

Lemma 4.18. *If non-empty, the space $\mathcal{S}_g^\mathfrak{a} \subset \mathcal{S}_g^{\times m}$ is irreducible.*

Proof. Recall from the introduction of Chapter 4 that this problem, when solved for $\mathbb{C}$, implies the result for all algebraically closed fields of char $\neq 2$. Over $\mathbb{C}$ we reduce the monodromy problem to counting $\mathrm{Sp}(V)$-orbits of m-tuples of quadratic forms on $V = \mathrm{H}^1(X, \mathbb{Z})$ as proven in Lemma 4.2. We proved in Proposition 4.9 that the $\mathrm{Sp}(V)$-orbit of an m-tuple of quadratic forms is completely determined by the Arf invariants of individual forms and the sum of triplets. The exponent $\mathfrak{a}$ in $\mathcal{S}_g^\mathfrak{a}$ fixes exactly this set of invariants and thus $\mathcal{S}_g^\mathfrak{a}$ is irreducible. $\square$

Remark 4.19. We will often consider triplets of theta characteristics of fixed parities (a_1, a_2, a_3). Then the corresponding moduli space has two irreducible components: one where the triplets are syzygetic, $a_{23} = 1$, and the other where the triplets are asyzygetic, $a_{23} = 0$.

There are $2^{m+\binom{m-1}{2}}$ choices of $\mathfrak{a}$ but not all of them will be realized. But we will prove that they will all be realized provided $g \geq m$.

Example 4.20. Let $\mathfrak{a} = (1, 1)$, i.e., we are considering the moduli spaces of pairs of odd theta characteristics. Then $\mathcal{S}_1^{(1,1)} = \emptyset$. Indeed, an elliptic curve has only one odd theta characteristic.

Example 4.21. Let $\mathfrak{a} = (1, 1, 1; 1)$, i.e., we are considering the moduli space of syzygetic triples of odd theta characteristics. Then $\mathcal{S}_2^{\mathfrak{a}} = \emptyset$. To see this, take a genus 2 hyperelliptic curve C with Weierstrass points $w_1, \ldots, w_6$. Up to relabeling, 3 distinct odd theta characteristics can be written as $\eta_i \equiv w_i$ for $i = 1, 2, 3$. But then $\eta_{123} := \omega_C^{\otimes 2} \otimes \eta_1^{-1} \otimes \eta_2^{-1} \otimes \eta_3^{-1} \equiv w_1 + w_2 + w_3 - g_2^1$, which is an even theta characteristic. Thus, any distinct triple of odd theta characteristics on C are asyzygetic.

Theorem 4.22. *The irreducible non-degenerate components $\mathcal{S}_g^{\mathfrak{a}}$ of $\mathcal{S}_g^{\times m}$ for all $\mathfrak{a} \in \mathbb{F}_2^{m+\binom{m-1}{2}}$ are non-empty iff $g \geq m$. In particular, $\mathcal{S}_g^{\times m}$ has precisely $2^{m+\binom{m-1}{2}}$ non-degenerate components iff $g \geq m$.*

Proof. In light of Lemma 4.18, and from the definition of $\mathcal{S}_g^{\mathfrak{a}}$, it is clear that any irreducible non-degenerate component of $\mathcal{S}_g^{\times m}$ must be of the form $\mathcal{S}_g^{\mathfrak{a}}$. What remains to be proven is that all $\mathcal{S}_g^{\mathfrak{a}}$ are non-empty iff $g \geq m$. This follows from Section 4.2.1. In particular, Corollary 4.13 implies that *all* $\mathcal{S}_g^{\mathfrak{a}}$ are non-empty if $g \geq m$. For the converse, pick $\mathfrak{a}$ to be the sequence consisting of just 1's. Applying Proposition 4.12 to this $\mathfrak{a}$ implies that $\mathcal{S}_g^{\mathfrak{a}}$ must be empty when $g < m$. $\square$

Remark 4.23. Suppose $g < m$ and $\mathfrak{a} \in \mathbb{F}_2^{m+\binom{m-1}{2}}$. As we have seen in the proof above, Proposition 4.12 provides an effective algorithm to check whether or not $\mathcal{S}_g^{\mathfrak{a}}$ is empty. From $\mathfrak{a}$ construct $\mathfrak{c}$ by the elementary operations given in Section 4.2.1 which also gives us the symmetric matrix E. We set $\dim V = 2g$, $\dim W = m - 1$ and we can compute c by inspection. It remains to compute $\dim(\ker E)$ in order to check the inequality of Proposition 4.12. If the inequality does not hold then $\mathcal{S}_g^{\mathfrak{a}}$ is empty and non-empty otherwise.

Example 4.24. When $m = 2$ and $g \geq 2$ then $\mathcal{S}_g^{\times m}$ has precisely 4 non-degenerate components: pairs of theta characteristics with parities (1,1),(1,0),(0,1),(0,0). We denote these by $\mathcal{S}_g^{--}, \mathcal{S}_g^{-+}, \mathcal{S}_g^{+-}, \mathcal{S}_g^{++}$ respectively. Of course, $\mathcal{S}_1^{\times 2}$ has only three non-degenerate components.

Example 4.25. When $m = 3$ and $g \geq 3$ then $\mathcal{S}_g^{\times m}$ has precisely 16 non-degenerate components. These components are determined by the parity of the three theta characteristics and whether or not they are syzygetic or asyzygetic.

4.3.1 On roots of the twisted canonical bundle

The proof of the following two results does not rely on our arguments but those of Jarvis [Jar00]. However, the line of reasoning is similar to what we have done so far. Recall that $\mathcal{S}^2_{g,2}(\omega_\pi(\sigma_1 + \sigma_2))$ parametrizes marked curves (C, p, q) together with a pair of distinct roots $\tau_1, \tau_2 \in \sqrt{\omega_C(p + q)}$.

Proposition 4.26. *The moduli space $\mathcal{S}^2_{g,2}(\omega_\pi(\sigma_1 + \sigma_2))$ is irreducible.*

Proof. This time the base is $\mathcal{M}_{g,2}$. Over a marked curve (C, p, q) we would like to know the monodromy action on the elements of $\sqrt{\omega_C(p + q)}$. This is precisely the content of Lemma 3.4 of [Jar00]. Identifying the set $\sqrt{\omega_C(p + q)}$ non-canonically with $V = \mathrm{H}^0(C, \mathbb{F}_2)$, Lemma 3.4 *loc. cit.* applied to our particular situation implies that the monodromy group is an extension of $\mathrm{Sp}(V)$ with the group of translations V. Therefore, when acting on pairs of roots we can, essentially, translate the first root and then act via the spin group on the remaining root. This proves that the monodromy action of $\mathcal{S}^2_{g,2}(\omega_\pi(\sigma_1 + \sigma_2)) \to \mathcal{M}_{g,2}$ is transitive on the fibers. $\square$

Corollary 4.27. *The boundary component Δ_0^{nn} as well as the boundary components Δ_0^{bn} and Δ_0^{nb} of $\overline{\mathcal{S}}_g^{--}$ are irreducible.*

Proof. The boundary Δ_0^{nn} is the image of $\mathcal{S}^2_{g,2}(\omega_\pi(\sigma_1 + \sigma_2))$ under the clutching map (see Section 1.7.1). Proposition 4.26 implies that the domain of this clutching map is irreducible and therefore, that Δ_0^{nn} is irreducible.

For Δ_0^{bn} and Δ_0^{nb} the argument is almost identical. In order to show that the domain of the relevant clutching map is irreducible, we have to argue as in the proof of Proposition 4.26, translating the root of the twisted canonical bundle and acting via the spin group on the odd theta characteristic. $\square$

Remark 4.28. In fact, the proof of Lemma 3.4 in [Jar00] contains an error. Halfway through the proof, two loops on a surface are taken with the assumption that their sum is homologous to zero. Later, the computations rely on their *difference* being homologous to zero. Theorem 3.1 of [Jar00] relies on this lemma and it would be interesting to know how the statement of this theorem has to be altered, if at all.

Nevertheless, this problem does not impact our application. When working with *square* roots, these computations are done in characteristic 2 and any sign error vanishes. In particular, the statement as we have used it is correct.

Part III

Appendicies

Chapter 1

Basics on hyperelliptic curves

1.1 Theta characteristics on hyperelliptic curves

Let C be a hyperelliptic curve of genus $g \geq 1$, $W = \{w_1, \dots, w_{2g+2}\}$ the set of Weierstrass points of C and $\mathfrak{g}^1_2$ the degree 2 pencil on C. For any $I \subset [2g+2]$ we will denote by w_I the divisor $\sum_{i \in I} w_i$.

Lemma 1.1. *Let $\mathfrak{w} = w_1 + \cdots + w_{2g+2}$ be the sum of all Weierstrass points. Then $\mathfrak{w} \in |(g+1)\mathfrak{g}^1_2|$. Therefore, for any $I \subset [2g+2]$ we have $w_I \equiv w_{I^c} + (\#I - (g+1))\mathfrak{g}^1_2$ and in particular when $\#I = g+1$ we have $w_I \equiv w_{I^c}$.*

Proof. Let $\pi : C \to \mathbb{P}^1$ be the double covering map. We have the Riemann–Hurwitz exact sequence:

$$0 \longrightarrow \pi^*\Omega_{\mathbb{P}^1} \longrightarrow \Omega_C \longrightarrow \Omega_{C/\mathbb{P}^1} \longrightarrow 0$$

Now, putting together $\pi^*\mathcal{O}_{\mathbb{P}^1}(1) \equiv \mathfrak{g}^1_2$, $\Omega_{\mathbb{P}^1} \simeq \mathcal{O}_{\mathbb{P}^1}(-2)$, $\Omega_C \equiv (g-1)\mathfrak{g}^1_2$ and $\Omega_{C/\mathbb{P}^1} \simeq \mathcal{O}_{\mathfrak{w}}$ we see that $\mathfrak{w} \in |(g+1)\mathfrak{g}^1_2|$.

For any $I \subset [2g+2]$ we have $w_I + w_{I^c} = \mathfrak{w} \equiv (g+1)\mathfrak{g}^1_2$. Subtracting w_{I^c} from both sides we get $w_I \equiv w_{I^c} + (g+1-\#I^c)\mathfrak{g}^1_2$. Plugging in $\#I^c = 2g+2-\#I$ gives us the desired formula. □

Lemma 1.2. *For any $0 \leq i \leq g$ and any $I \in \binom{[2g+2]}{i}$ we have $h^0(w_I) = 1$. If $I \subset [2g+2]$ has $g+1$ elements then $h^0(w_I) = 2$ and $|w_I|$ is base point free.*

Proof. Let $I \subset [2g+2]$ be of size g. Then $h^0(w_I) = 1 + h^0(\omega_C - w_I)$. Since there is no hyperplane intersecting the rational normal curve in $\mathbb{P}^{g-1}$ through g distinct points $h^0(\omega_C - w_I) = 0$. Since the sum of g distinct Weierstrass points do not move, neither can the sum of fewer Weierstrass points.

Now suppose I has $g+1$ points. We know that $w_{I^c} \in |w_I|$ so that $h^0(w_I) \geq 2$. But since g Weierstrass points have only 1 section and number of sections can only increase by 1 per degree we must have $h^0(w_I) = 2$. By what has gone before, this must be base point free. □

Definition 1.3. Given $I \subset [2g+2]$ with $i \equiv g-1 \pmod 2$ we define a *theta characteristic* η_I on C as follows. Let $u = \frac{g-1-i}{2}$ and $\eta_I = w_I + u\mathfrak{g}^1_2$. Clearly $\eta_I^{\otimes 2} \simeq \omega_C$.

Lemma 1.4. *With η_I and u defined as above we have $h^0(\eta_I) = |u+1|$.*

Proof. If $u+1 < 0$ then $i = \#I > g+1$. Using Lemma 1.1 we can take the complement of I without changing the isomorphism class of η_I. This allows us to assume $i \leq g+1$ and $u+1 \geq 0$ for the remainder of the proof.

Since $2\eta_I \equiv \omega_C$ we have $h^0(\eta_I) = h^0(\omega_C - \eta_I)$. For $i < g$, and hence $u \geq 0$, we can apply geometric Riemann–Roch to see that $h^0(\omega_C - w_I - u\mathfrak{g}^1_2) - 1$ is the dimension of the space of hyperplanes intersecting the canonical image of C in u fixed (general) points and in the images of w_I. This gives $u+i$ distinct points on the canonical curve, thus there is a $g-u-i-1$ dimensional space of such hyperplanes. Since $2u+i = g-1$ we conclude that $h^0(\eta_I) = u+1$.

By parity reasons we can not have $i = g$. When $i = g+1$ we saw that w_I is a base point free $\mathfrak{g}^1_{g+1}$ and so $w_I - \mathfrak{g}^1_2$ can not move, i.e., $h^0(\eta_I) = 0 = u+1$. □

Lemma 1.5. *If $\eta_I \simeq \eta_J$ then either $I = J$ or $I = J^c$.*

Proof. Suppose $I \neq J, J^c$ but $\eta_I \simeq \eta_J$ to derive a contradiction. Let $i = \#I$ and $j = \#J$ and note $i \equiv j \pmod 2$. We have $w_I + u\mathfrak{g}^1_2 \equiv w_J + v\mathfrak{g}^1_2$ iff $w_I + w_J \equiv t\mathfrak{g}^1_2$ for $t = \frac{i+j}{2}$. Unless $I \cap J = \emptyset$ we can remove from both sides some $\mathfrak{g}^1_2$'s and assume that $I \cap J = \emptyset$ anyway. This rules out $i = j = g+1$ and hence $i + j < 2g+2$.

If $i + j > g+1$ then subtract $w_I + w_J$ from $\mathfrak{w} \equiv (g+1)\mathfrak{g}^1_2$ and now we can assume $i + j \leq g+1$. But $w_{I\cup J}$ when $i+j \leq g+1$ can not be of the form $t\mathfrak{g}^1_2$ due to Lemma 1.2. □

Corollary 1.6. *Any theta characteristic on a hyperelliptic curve C is of the form $\eta_I = w_I + u\mathfrak{g}^1_2$.*

Proof. Since $\operatorname{Jac} C$ is a g-dimensional torus, we know that there will be 2^{2g} roots of the canonical bundle, up to isomorphism. We will now count the number of η_I's that are possible. We need $I \in \binom{[2g+2]}{i}$ for some $i \equiv g-1 \pmod 2$. Half the subsets of any set are of one parity, the other half is of the other parity (fix a point and remove or add it to establish this bijection). Moreover, we know that $\eta_I \simeq \eta_{I^c}$ so that by Lemma 1.5 there are precisely $\frac{1}{4} \cdot \#\mathcal{P}([2g+2]) = 2^{2g}$ non-isomorphic η_I's. Since the two numbers agree, all theta characteristics are of the form η_I. □

Given a triplet (η_I, η_J, η_K) of distinct odd theta characteristics we will often want to say wether or not this triplet is syzygetic. The following lemma gives an easy formula to compute this in the cases we will need.

Lemma 1.7. *Let $I, J, K \subset [2g+2]$ be distinct subsets of size $g-1$. Then $h^0(\eta_I \otimes \eta_J \otimes \eta_K \otimes \omega_C^{-1}) \equiv \#I \cap J + \#I \cap K + \#J \cap K - g \pmod 2$.*

Proof. Let $a_{IJ} = \#I \cap J$, $a_{IK} = \#I \cap K$, $a_{JK} = \#J \cap K$ and $a_{IJK} = \#I \cap J \cap K$. Let $L = I+J+K$ with the usual addition operation on the power set $\mathcal{P}([2g+2]) \simeq \mathbb{F}_2^{2g+2}$. It is easy to see that $\#L = 3g-3-2(a_{IJ}+a_{IK}+a_{JK}-2a_{IJK})$. Clearly $\eta_L \simeq \eta_I \otimes \eta_J \otimes \eta_K \otimes \omega_C^{-1}$ and so we may use Lemma 1.4 to get $h^0(\eta_L) = |a_{IJ} + a_{IK} + a_{JK} - 2a_{IJK} - g + 2|$. This formula modulo 2 yields the claim. □

1.2 Associated theta markings on hyperelliptic curves

Our goal is to show that the moduli space of theta markings is irreducible. To do this we will specialize to the hyperelliptic locus and most of the proof will take place over this locus. Before we reduce to hyperelliptic curves, let us define the objects of interest in general and make some basic observations.

Fix $g \geq 3$ and let $\mathcal{S} = \{(C, \mu_1, \mu_2) \in \mathcal{S}_g^{--} \mid h^0(\mu_i) = 1\}$ such that $[\mu_1, \mu_2]$ has no base points. Define $\mathcal{P} = \{(C, \mu_1, \mu_2, , D \in [\mu_1, \mu_2])\}$ over $\mathcal{S}$. Over $\mathcal{P}$ we define $Z_1 = \{(C, \mu_1, \mu_2, , D \in [\mu_1, \mu_2], p \in C) \mid p \leq D\}$ and, in a way to be made precise later, we define $Z_2 = \{(C, \mu_1, \mu_2, , D \in [\mu_1, \mu_2], (p, q) \in C^2) \mid p + q \leq D\}$. Recall that $\mathcal{P} \to \mathcal{S}$ is a $\mathbb{P}^1$-bundle and $Z_i \to \mathcal{P}$ is finite for both $i = 1$ and $i = 2$.

Let us remark here that $\mathcal{S}$ is not representable by schemes because some curves and all theta characteristics have non-trivial automorphisms. However, the spaces $\mathcal{P}$ and Z_i's constructed above are representable over $\mathcal{S}$. Now we can state precisely our goal, whose proof will take the rest of the section.

Theorem 1.8. *The moduli stack Z_2 is irreducible.*

Let us first define our objects precisely. Let $\pi : \mathcal{C} \to \mathcal{S}$ the universal curve and $\mathcal{L}_1, \mathcal{L}_2$ the two theta characteristics over it. Let $\mathcal{N} = (\pi_* \mathcal{L}_1)^{\otimes 2} \oplus (\pi_* \mathcal{L}_2)^{\otimes 2}$ and notice that $\mathcal{N}$ is a rank-2 vector bundle. Moreover, there is an inclusion $\mathcal{N} \hookrightarrow \pi_* \omega_\pi$ obtained via the squaring maps $\mathcal{L}_i^{\otimes 2} \xrightarrow{\sim} \omega_\pi$.

We have $\mathcal{P} = \underline{\mathrm{Proj}}(\mathcal{N}^\vee)$ which parametrizes sub-bundles of rank-1 in $\mathcal{N}$. Let $\mathcal{W} \hookrightarrow \mathcal{N}$ be the universal sub-bundle over $\mathcal{P}$. Over $\rho : \mathcal{C}_\mathcal{P} := \mathcal{P} \times_\mathcal{S} \mathcal{C} \to \mathcal{P}$ we have a natural map $\rho^* \mathcal{W} \to \omega_\rho$, which cuts out an associated divisor on each fiber. Thus Z_1 is the subscheme cut as the zero locus of this map, or more precisely by the ideal sheaf $\mathrm{im}(\rho^* \mathcal{W} \otimes \omega_\rho^\vee \to \mathcal{O}_{\mathcal{C}_\mathcal{P}})$.

By construction of Z_1, if $(\pi : C \to B, L_1, L_2, W \hookrightarrow N, s : B \to C)$ is an object in $\mathcal{C}_\mathcal{P}$ then this object belongs to Z_1 iff $W = s^* \pi^* W \to s^* \omega_\pi$ is the zero map. This allows us to make the following observation:

Lemma 1.9. *The forgetful functor $Z_1 \to \mathcal{C}$ taking (C, μ_1, μ_2, D, p) to (C, μ_1, μ_2, p) is an isomorphism.*

Proof. Before we begin the proof, let us illustrate why this is expected. If we fix (C, μ_1, μ_2), having assumed $[\mu_1, \mu_2]$ is base point free, any point $p \in C$ will uniquely determine a divisor $D \in [\mu_1, \mu_2]$ for which $p \leq D$. This defines the inverse map $\mathcal{C} \to Z_1$ pointwise.

In order to construct the inverse $\mathcal{C} \to Z_1$ properly, we will construct a map $\mathcal{C} \to \mathcal{C}_\mathcal{P}$ and use the functorial interpretation of Z_1 given above.

With π and $\mathcal{N}$ defined above, on $\mathcal{C}$ we have $\pi^*\mathcal{N} \to \pi^*\pi_*\omega_\pi \to \omega_\pi$. The composition $\pi^*\mathcal{N} \to \omega_\pi$ is surjective because $\mathcal{N}$ is assumed to be base point free on fibers. Let $\mathcal{K} = \ker(\pi^*\mathcal{N} \twoheadrightarrow \omega_\pi)$, which is a line bundle on $\mathcal{C}$. This construction induces a map $\mathcal{C} \to \underline{\mathrm{Proj}}(\mathcal{N}^\vee) = \mathcal{P}$ via the universal property of $\mathcal{P}$. Then by the universal property of the product we get a map $\mathcal{C} \to \mathcal{C}_\mathcal{P} = \mathcal{P} \times_\mathcal{S} \mathcal{C}$.

The universal curve over $\mathcal{C}$ is $\mathcal{C}^2 := \mathcal{C} \times_\mathcal{S} \mathcal{C}$ with the universal section $\Delta : \mathcal{C} \to \mathcal{C}^2$ and projections $\mathrm{pr}_i : \mathcal{C}^2 \to \mathcal{C}$. We now check if the map $\mathcal{K} \to \Delta^*\omega_{\mathrm{pr}_1}$ is the zero map, for then the morphism $\mathcal{C} \to \mathcal{C}_\mathcal{P}$ will factor through Z_1. Indeed, we start with $\mathrm{pr}_1^*\mathcal{K} \to \mathrm{pr}_1^*\pi^*\mathcal{N} \to \mathrm{pr}_1^*\omega_\pi \to \omega_{\mathrm{pr}_1}$. Since $\omega_{\mathrm{pr}_1} \simeq \mathrm{pr}_2^*\omega_\pi$, when we pullback the sequence of morphism above with respect to Δ^* we get the zero map by definition of $\mathcal{K}$:

$$\mathcal{K} \to \pi^*\mathcal{N} \to \omega_\pi \xrightarrow{\mathrm{id}} \omega_\pi.$$

□

It follows from general principles that $Z_1 \times_\mathcal{P} Z_1 \simeq \mathcal{C} \times_\mathcal{P} \mathcal{C}$ is a closed substack of $(\mathcal{C} \times_\mathcal{S} \mathcal{C}) \times_\mathcal{S} \mathcal{P}$.

Definition 1.10. Define Z_2 as follows, where closure is to be taken in $(\mathcal{C} \times_\mathcal{S} \mathcal{C}) \times_\mathcal{S} \mathcal{P}$:

$$Z_2 := \overline{Z_1 \times_\mathcal{P} Z_1 \setminus \Delta_{Z_1}}.$$

Let $\mathcal{H} \hookrightarrow \mathcal{S}$ be the closed substack parametrizing tuples (C, μ_1, μ_2) where C is hyperelliptic. Let $\mathcal{C}' \to \mathcal{H}$ stand for the pullback of $\mathcal{C} \to \mathcal{S}$ to $\mathcal{H}$. Similarly define $\mathcal{P}' := \mathcal{P}|_\mathcal{H}$ and $Z_i' := Z_i|_\mathcal{H}$ for $i = 1, 2$. There is a universal involution $\iota : \mathcal{C} \to \mathcal{C}$ restricting to the hyperelliptic involution on each geometric fiber, see [LK79].

Lemma 1.11. *There is a section $\sigma : Z_1' \to Z_2'$ which after the identifications $Z_1' \simeq \mathcal{C}$ and $Z_2' \subset \mathcal{C} \times_{\mathcal{P}'} \mathcal{C}$ is simply $(\mathrm{id}, \iota) : \mathcal{C}' \to \mathcal{C}' \times_{\mathcal{P}'} \mathcal{C}'$.*

Proof. We expect this to be true by considering a single hyperelliptic curve C with involution $p \mapsto \bar{p}$. Every canonical divisor of C is invariant under conjugation and Weierstrass points always appear with multiplicity 2. Therefore, if $p \in C$ is such that $p \leq D \in [\mu_1, \mu_2] \subset |\omega_C|$ we must have $p + \bar{p} \leq D$. Pointwise this defines $Z_1' \to Z_2' : p \mapsto p + \bar{p}$.

When dealing with families, simply observe that $\iota : \mathcal{C}' \to \mathcal{C}'$ commutes with the map to $\mathcal{P}'$. Thus the graph of ι defines $\mathcal{C}' \to \mathcal{C}' \times_{\mathcal{P}'} \mathcal{C}'$ which does not factor through the diagonal. Identifying $\mathcal{C}'$ with Z_1' in light of Lemma 1.9 we obtain the desired section. □

Having identified one of the components of Z_2' as the image of σ we may define $Z_2'' \subset Z_2'$ to be the union of the rest of the components. We will prove Z_2'' is irreducible, but before we can prove this, we need to digress into rational normal curves.

Let $\mathcal{M}_{0,N}$ stand for the moduli space of genus 0 curves with N marked points and let Σ_m be the permutation group on m elements. With n and $m_1, \ldots, m_s$ non-negative integers adding up to N let $\mathcal{M}_{0,n,\{m_1\},\ldots,\{m_s\}}$ stand for the quotient $\mathcal{M}_{0,N}/\Sigma_{m_1} \times \cdots \times \Sigma_{m_s}$ where Σ_{m_j} acts on the markings in the range $(n + 1 + \sum_{i=1}^{j-1} m_i)$ to $(n + \sum_{i=1}^{j} m_i)$.

The canonical map realizes a genus g hyperelliptic curve C as a 2:1 cover of the $g-1$-th Veronese embedding of $\mathbb{P}^1$. The $2g+2$ Weierstrass points of C map to $2g+2$ points of $\mathbb{P}^1$ and each rigid odd theta characteristics η on C distinguishes a subset of the Weierstrass points of size $g-1$, and thus of the points of $\mathbb{P}^1$.

Let (C, η_1, η_2) be a hyperelliptic curve with two distinct rigid and odd theta characteristics. This gives $2g+2$ points on $\mathbb{P}^1$ with these points partitioned into sets of size $(g-1, g-1, 4)$. Therefore, if we define $B = \mathcal{M}_{0,\{g-1\},\{g-1\},\{4\}}$ we get a natural map $\mathcal{S} \to B$.

Let $r_1, \ldots, r_{g-1}$ and $r'_1, \ldots, r'_{g-1}$ stand for the points on $\mathbb{P}^1$ distinguished by η_1 and η_2 respectively. The pencil of associated divisors $[\eta_1, \eta_2]$ on C is simply the pullback of the pencil of divisors spanned by $\overline{r} := r_1 + \cdots + r_{g_1}$ and $\overline{r}' := r'_1 + \cdots + r'_{g-1}$ in $|\mathcal{O}_{\mathbb{P}^1}(g-1)|$. Denote this pencil by $[\underline{r}, \underline{r}'] \subset |\mathcal{O}_{\mathbb{P}^1}(g-1)|$.

Writing $B = \{(\mathbb{P}^1, \underline{a}, \underline{b}, \underline{c})\}$ we define a $\mathbb{P}^1$-bundle over B as $P = \{(\mathbb{P}^1, \underline{a}, \underline{b}, \underline{c}, D) \mid D \in [\underline{a}, \underline{b}]\}$. Finally we define $W := \{(\mathbb{P}^1, \underline{a}, \underline{b}, \underline{c}, D, r, r') \mid r + r' \leq D \in [\underline{a}, \underline{b}]\}$, which is a simpler version of Z''_2.

Lemma 1.12. *The space W is irreducible.*

Proof. Since B is irreducible, so is the $\mathbb{P}^1$-bundle $P \to B$. Moreover, $W \to P$ is finite. Therefore, it remains to construct an irreducible subspace in W containing an entire fiber of $W \to P$ over a non-branch point.

Define a section $\sigma : B \to P$ by choosing $D = \underline{a}$. Pullback W via σ to get $W_B \to B$. It is easy to see that $W_B \simeq \mathcal{M}_{0,2,\{g-3\},\{g-1\},\{4\}}$, which is irreducible. Since the image of σ contains non-branch points of $W \to P$ the subscheme $W_B \hookrightarrow W$ satisfies the desired criteria. $\square$

We described a map $\mathcal{S} \to B$, and clearly this induces a map $Z''_2 \to W$ which is finite of degree 4. Indeed, fixing $C \xrightarrow{2:1} \mathbb{P}^1$, and the marked points $(r, r') \in \mathbb{P}^1$ with $p \mapsto r$ and $p' \mapsto r'$, the fiber of $Z''_2 \to W$ over (r, r') consists of the following 4 markings: $(p, p'), (p, \bar{p}'), (\bar{p}, p'), (\bar{p}, \bar{p}')$ with bar denoting conjugation as usual.

For the following results, we will give a proof over $\mathbb{C}$ for readability. The results hold over any base field $k = \bar{k}$ with $\operatorname{char} k \neq 2$ but one has to argue as in the proof of Lemma 1.12.

Lemma 1.13. *The space Z''_2 is irreducible.*

Proof. We just need to show the monodromy action $Z''_2 \to W$ is transitive. To that end, fix $(C, \eta_1, \eta_2) \in \mathcal{S}$ which corresponds to $(\mathbb{P}^1, \underline{a}, \underline{b}, \underline{c}) \in B$. Let $T = [\underline{a}, \underline{b}] \simeq \mathbb{P}^1$, pick a reduced divisor $D \subset T$ and $r + r' \subset D$. As before we denote the preimages of r and r' in C by $p, \bar{p}$ and $p', \bar{p}'$ respectively.

On T construct a path γ as follows: the image of γ should avoid all ramification points, it should start at D and move very close to $\underline{a}$, go around $\underline{a}$ once and trace back the initial path back to D. If the loop around $\underline{a}$ is small enough, this path does not permute the fibers of $W|_T$. However, since a_i's are the branch points of $C \xrightarrow{2:1} \mathbb{P}^1$, monodromy via this loop will conjugate marked points. In particular, this shows that (p,p') and $(\bar{p},\bar{p}')$ lie on the same component of Z_2'', and $(\bar{p},p')$ and $(p,\bar{p}')$ lie on the same component of Z_2''.

It remains to connect (p,p') with $(\bar{p},p')$. The idea is as follows: find a divisor $E \in T$ such that E is reduced, and contains one of the branch points of $C \to \mathbb{P}^1$. Note that we have complete freedom in choosing $\underline{a}$ and $\underline{b}$ to begin with, which determines the pencil T. Therefore, the existence of such an E is not an issue. Now we take a small loop around E and connect it to D in T while avoiding non-reduced divisors. This path will act trivially on $W|_T$ but will have the effect of interchanging a pair of conjugate points in C mapping to D. By choosing a_i appropriately, we can arrange it so that p and $\bar{p}$ are interchanged, while p' is fixed. This path then lifts to a path between (p,p') and $(\bar{p},p')$.

This shows that the monodromy of $Z_2'' \to W$ acts transitively on the 4 points over (r,r'). Hence Z_2'' is irreducible. □

We are now ready to prove that Z_2 itself is irreducible. Before we begin the proof, let us outline the strategy. The subspaces Z_2' and Z_2'' of Z_2 are irreducible and they will contain a fiber of $Z_2 \to \mathcal{P}$ without ramification points. Therefore, Z_2 can have at most 2 components: the one containing Z_2' and the one containing Z_2''. Our goal is then to construct a monodromy action of $Z_2 \to \mathcal{P}$ sending a point in Z_2' to a point in Z_2''.

Proof of Theorem 1.8. Note that Z_2' contains markings of the form (w_i,w_i) for a Weierstrass point w_i but not of the form (w_i,w_j) if $i \neq j$. Conversely, Z_2'' contains points of the form (w_i,w_j) when $i \neq j$ but not if $i = j$.

We will work with a non-hyperelliptic curve C and pick $(C,\eta_1,\eta_2) \in \mathcal{S}$. If $\eta_1 = \sum_i p_i$ and $\eta_2 = \sum_i q_i$ then, for $i = j$ and $i \neq j$, (p_i,p_j) will be a marking in Z_2 since $2\sum_i p_i \in [\eta_1,\eta_2]$. Clearly, if we specialize C to a hyperelliptic curve, (p_i,p_j) will specialize to (w_i,w_j). We can choose markings that are close to (p_i,p_j) if we wish to specialize to $(p,\bar{p})$ when $i = j$ and to (p,p') when $i \neq j$.

Let $T = [\eta_1,\eta_2]$ and $R := Z_2|_T$. Lemma 1.9 implies that $Z_1|_T \simeq C$ and the natural map $Z_1|_T \to T$ induces a covering map $C \to T \simeq \mathbb{P}^1$. Since $R = \overline{C \times_T C \setminus \Delta_C}$, to describe the monodromy of $R \to T$ it suffices to understand the monodromy of $C \to T$.

Clearly $C \to T$ has ramification profile of type (2^{g-1}) over $2\eta_1$ and $2\eta_2$. By degenerating to hyperelliptic curves and studying rational normal curves, it can be seen that the branching elsewhere is simple, that is of the form $(2,1^{2g-4})$. Here we use the fact that C is general. Choose a reduced divisor $D' \in T$ and choose a small loop around $2\eta_1$ and connect this small loop to D'. This path induces a permutation action τ on the points of D' and we label the points of D' so that $D' = \sum_{i=1}^{g-1}(p_i' + p_i'')$ where $\tau(p_i') = p_i''$ and $\tau(p_i'') = p_i'$. Let ψ be another permutation action on the support of D' induced by one of the simple branches of $C \to T$. Since C is irreducible, we can take one ψ which, up to relabeling of points, interchanges p_1' and p_2' and acts trivially otherwise.

We have just shown that there is a monodromy action of $R \to T$ mapping (p_1',p_1'') to (p_2',p_1''). Specializing C to hyperelliptic curves and using the argument in the first paragraph of this proof, we see that this monodromy interchanges a point in Z_2' with a point in Z_2''. Therefore Z_2 is irreducible. □

Chapter 2

Geometric interpretation

The main purpose of this appendix is to explain how torsion-free sheaves of rank-1 and blow-ups of curves are really the same thing. We then recall how limit roots, as defined in [CCC07], agree with the (torsion-free) roots of [Jar98]. This is done in order to explain the underlying geometric intuition behind our definition for multiple roots.

2.1 Blow-ups of curves

Let k be an algebraically closed field with $\operatorname{char} k \neq 2$. Let C be a stable curve over k and N a line bundle on C with $\deg N$ divisible by 2.

Remark 2.1. We don't need such strong hypotheses on k. Most of what we say in this appendix will work if the word *node* is replaced with *split node*. The rest will work if $\operatorname{char} k \neq 2$ and $\sqrt{k} = k$. But we will not use this fact.

Suppose $x_1, \dots, x_v \in C$ are some of the nodes of C. Let $I := I(\mathfrak{x}) \subset \mathcal{O}_C$ be the ideal sheaf corresponding to the subscheme $\mathfrak{x} := \{x_1, \dots, x_v\} \subset C$. Define $\pi : X = \underline{\operatorname{Proj}}_C(\operatorname{Sym}^* I) \to C$. Observe that π is an isomorphism in the open set $C \setminus \mathfrak{x}$. And for any $x \in \mathfrak{x}$, the fiber of $X \to C$ over x is isomorphic to $\mathbb{P}^1_k$. This motivates us in making the following abuse of notation:

Definition 2.2. For a subset of the nodes $\mathfrak{x} \subset C$, the corresponding construction $\pi : X \to C$ given above will be called a *blow-up* of C at $\mathfrak{x}$. Denote this object by $\pi : \operatorname{Bl}_{\mathfrak{x}} C \to C$ and allow for $\mathfrak{x} = \emptyset$. If X is the blow-up of a stable curve then X is called a *quasi-stable curve*.

Remark 2.3. This is indeed an abuse of notation. The actual blow-up of C at $\mathfrak{x}$ is defined via the Rees algebra and not the symmetric algebra of $I(\mathfrak{x})$. So the actual blow-up construction partially normalizes C at $\mathfrak{x}$. The canonical surjective map $\operatorname{Sym}^*(I) \to \operatorname{Rees}(I)$ induces a closed immersion from the partial normalization of C at $\mathfrak{x}$ into the blow-up.

Remark 2.4. The $\underline{\operatorname{Proj}}$ construction yields, in addition, a line bundle on the total space. Only with this line bundle is this object truly meaningful, i.e., it satisfies a universal property (see [Stacks, Tag 01NS]). Indeed if we were to consider blow-ups together with their defining line bundles, we would go full circle and recognize that these objects are in correspondence with torsion-free sheaves of rank-1.

Definition 2.5. Given any $Y \to C$ if there is a subset $\mathfrak{x} \subset C$ of the nodes and an isomorphism

$$\begin{tikzcd} Y \arrow[rr, "\sim"] \arrow[dr] & & \mathrm{Bl}_{\mathfrak{x}} C \arrow[dl] \\ & C & \end{tikzcd}$$

then we will also call $Y \to C$ a blow-up of C at $\mathfrak{x}$.

Definition 2.6. Let $\pi : X \to C$ be a blow-up of the nodes $\mathfrak{x} \subset C$. Then for each $x \in \mathfrak{x}$ we will call the fiber $\pi^{-1}(x) \subset X$ an *exceptional component* of X.

2.2 Limit roots on curves

We now recall the notion of a limit root given in Definition 2.1.1 of [CCC07].

Definition 2.7. Consider a triplet $(\pi : X \to C, L, \alpha : L^{\otimes 2} \to \pi^* N)$ where π is a blowup and L is a line bundle on X of degree $\frac{\deg N}{2}$. This triple is called a *limit root* of N if the following are satisfied:

- L has degree 1 on each exceptional divisor of π.
- α is an isomorphism in the complement of the exceptional components of X.

Remark 2.8. We get to ommit condition (iii) in Definition 2.1.1 of [CCC07] because we specified the degree of L and because we are considering *square* roots.

2.3 Families of limit roots

Let $\mathcal{C} \to T$ be a family of stable curves over a scheme T, where T is defined over $\mathbb{Z}[\frac{1}{2}]$. Let $\mathcal{N}$ be a line bundle on $\mathcal{C}$ of relative degree d, which we assume to be even. The following three definitions are from [CCC07].

Definition 2.9. Suppose that $\pi : \mathcal{X} \to \mathcal{C}$ is a morphism such that, $\mathcal{X} \to T$ is a family of nodal curves and for each geometric point $t \to T$ the fiber $\pi_t : \mathcal{X}_t \to \mathcal{C}_t$ is a blowup in the sense of Definition 2.2. Then we will call $\pi : \mathcal{X} \to \mathcal{C}$ a *family of blow-ups.*

Definition 2.10. Let $\pi : \mathcal{X} \to \mathcal{C}$ be a family of blowups, $\mathcal{L}$ a line bundle on $\mathcal{X}$ and $\alpha : \mathcal{L}^{\otimes 2} \to \pi^*\mathcal{N}$ a morphism. If $(\mathcal{X} \xrightarrow{\pi} \mathcal{C}, \mathcal{L}, \alpha)$ restricts on each geometric fiber to a limit root as in Definition 2.7, then we will call $(\mathcal{X} \xrightarrow{\pi} \mathcal{C}, \mathcal{L}, \alpha)$ a *family of limit roots.*

Definition 2.11. For $i = 1, 2$ let $(\pi_i : \mathcal{X}_i \to \mathcal{C}, \mathcal{L}_i, \alpha_i)$ be two families of limit roots. An isomorphism between them is a pair (f, g) where $f : \mathcal{X}_1 \to \mathcal{X}_2$ is an isomorphism over $\mathcal{C}$ and $g : \mathcal{L}_1 \to f^*\mathcal{L}_2$ is an isomorphism such that $\alpha_1 = f^*\alpha_2 \circ g^{\otimes 2}$.

We can now define the moduli space corresponding to limit roots.

Definition 2.12. Let $\overline{\mathcal{S}}(\mathcal{N})' \to T$ be the category fibered in groupoids, associating to each $T' \to T$ the groupoid of families of limit roots of $\mathcal{N}|_{T'}$ over $\mathcal{C}|_{T'} \to T'$.

2.3.1 Relation to torsion-free roots

Suppose $(\mathcal{E}, b)$ is a root of $\mathcal{N}$ on $\mathcal{C}$. Define $\mathbb{P}(\mathcal{E}) := \underline{\mathrm{Proj}}_{\mathcal{C}}(\mathrm{Sym}^* \mathcal{E})$ with $\pi : \mathbb{P}(\mathcal{E}) \to \mathcal{C}$ the structure map and $\mathcal{L} = \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ the line bundle corresponding to the $\underline{\mathrm{Proj}}$ construction. Notice that π is a family of blow-ups.

There are natural surjective maps $\pi^*\mathcal{E}^d \to \mathcal{L}^{\otimes d} = \mathcal{O}_{\mathbb{P}(\mathcal{E})}(d)$ for each $d \geq 0$ and there is the map $\pi^*b : \pi^*\mathcal{E}^2 \to \pi^*\mathcal{N}$. As is shown in §3.1.3 of [Jar98] there is a natural map α making the following diagram commute:

$$\begin{tikzcd}
\pi^*\mathcal{E}^2 \arrow[d] \arrow[dr, "\pi^*b"] & \\
\mathcal{L}^2 \arrow[r, dotted, "\alpha"] & \pi^*\mathcal{N}
\end{tikzcd}.$$

Proposition 2.13. *Let $(\mathcal{E}, b)$ be a root of $\mathcal{N}$. Then $(\mathbb{P}(\mathcal{E}) \xrightarrow{\pi} \mathcal{C}, \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1), \alpha)$, constructed above, is a family of limit roots of $\mathcal{N}$.*

Proof. Both $\underline{\mathrm{Proj}}$ and Sym constructions behave well with respect to base change. So we may reduce to $T = \operatorname{Spec} k$ where k is an algebraically closed field.

Let $L = \mathcal{O}(1)$ and note $\pi_* L \simeq \mathcal{E}$, see Lemma 3.1.4.(2) [Jar98]. To see that L has degree one over any exceptional fiber E over a node x we simply observe $h^0(E, L|_E) = \dim_k \mathcal{E}|_x = 2$. Since $E \simeq \mathbb{P}^1_k$ we are done.

The map α is an isomorphism away from the exceptional divisors because b is an isomorphism away from the corresponding nodes. The degrees of L and $\mathcal{E}$ agree because $\pi_* L \simeq \mathcal{E}$. This completes the proof. □

Conversely, given a family of limit roots $(\pi : \mathcal{X} \to \mathcal{C}, \mathcal{L}, \alpha)$, let $\mathcal{E} := \pi_*\mathcal{L}$. Then, using Lemma 3.1.4.(2) [Jar98] again, we have $\pi_*\mathcal{L}^2 \simeq \mathcal{E}^2$. Using the adjunction map $a : \pi_*\pi^*\mathcal{N} \to \mathcal{N}$ we may define $b := a \circ \pi_*\alpha : \mathcal{E}^2 \to \pi_*\pi^*\mathcal{N} \to \mathcal{N}$.

Proposition 2.14. *The tuple $(\mathcal{E}, b)$ obtained in this way is a root of $\mathcal{N}$*

Proof. This is similar to the proposition above. The main ingredients are Proposition 3.1.2.(3) and Proposition 3.1.5 of [Jar98] which says that $\pi_*\mathcal{L}$ is torsion-free and b is of the right form respectively. □

In summary, we conclude that families of limit roots (using quasi-stable curves) and families of roots (using torsion-free sheaves) are in fact equivalent notions. More precisely we may sate:

Corollary 2.15. *The categories $\overline{\mathcal{S}}(\mathcal{N})'$ and $\overline{\mathcal{S}}(\mathcal{N})$ are equivalent.*

Proof. The two propositions above constructs the functors which are clearly quasi-inverses to one another. The fact that these constructions behave functorially follows from the functorial behavior of of $\underline{\mathrm{Proj}}$ and pushforward. □

2.4 Multiple limit roots

Let $\mathcal{C} \to T$ be a stable curve. We want to consider m-tuples of limit roots of $\mathcal{N}$ over $\mathcal{C}$. Before we give our definition, let us explore some of the difficulties and dead ends.

2.4.1 Motivation

It will be sufficient to assume $m = 2$. Let us consider $\mathcal{Y} := \overline{\mathcal{S}}(\mathcal{N})' \times_T \overline{\mathcal{S}}(\mathcal{N})'$ for a moment. If this moduli space were to work, we could have stopped right here. But the main problem with $\mathcal{Y}$ can already be seen at its geometric points.

Let $t : \operatorname{Spec} k \to T$ be a geometric point. Let $C = \mathcal{C}|_t$ and $N = \mathcal{N}|_t$. If $\operatorname{Spec} k \xrightarrow{y} \mathcal{Y}$ is a geometric point lying over t, then y corresponds to two limit roots of N over C. We will denote these by $(X_i \xrightarrow{\pi_i} C, L_i, \alpha_i)$ for $i = 1, 2$.

Let us consider a particularly simple case. Suppose that π_i's are both blow-ups of a single node $x \in C$. Then the two curves X_1 and X_2 are isomorphic, but not canonically: The exceptional divisors $E_i := \pi_i^{-1}(x)$ are not uniquely identified with one another. There is a k^*-torsor of such isomorphisms between the X_i's.

We could ask for an isomorphism $f : X_1 \xrightarrow{\sim} X_2$ over C such that $f^* L_2^{\otimes 2} \simeq L_1^{\otimes 2}$ over $\pi_1^* \mathcal{N}$. This restriction is natural as we are asking for the two limit roots to square to the same line bundle. Moreover, there are now only two possible isomorphisms between the X_i's. Nevertheless, this is still not unique.

This demonstrates the main shortcoming of the fiber product $\mathcal{Y} = \overline{\mathcal{S}}(\mathcal{N})' \times_T \overline{\mathcal{S}}(\mathcal{N})'$. The tuples of limit roots can not be viewed as line bundles on a single family of curves. Hence the objects parametrized by $\mathcal{Y}$ are rather awkward. We summarize this here.

Proposition 2.16. *Objects of $\overline{\mathcal{S}}^m(\mathcal{N}) \times_{\mathcal{M}} \overline{\mathcal{S}}^m(\mathcal{N})$ are not geometrically meaningful.*

Proof. Recall $\overline{\mathcal{S}}^m(\mathcal{N}) \simeq \overline{\mathcal{S}}(\mathcal{N})'$ so that we reduce to the argument above. Objects in $\mathcal{Y}$ require two different curves and line bundles on them, this is not when wanting to compare the line bundles. □

The following fix to our problem readily suggests itself. If X_i is a blow-up of $\mathfrak{x}_i \subset C$ then let us fix a blow-up $\pi : X \to C$ of $\mathfrak{x} = \mathfrak{x}_1 \cup \mathfrak{x}_2$. The blow-up π non-canonically factors through π_i's so we must incorporate this factorization into our data to remove any ambiguity. In other words, we want to fix $\rho_i : X \to X_i$ such that $\pi = \pi_i \circ \rho_i$. If this data is available, L_i's can be canonically pulled back to X so that they share the same curve.

There is one fatal flaw with this approach however. Namely that there are way too many ρ_i's to give a good moduli space. Recall that when X_i's were the blow-ups of a single point $x \in C$, the tuples (ρ_1, ρ_2) up to isomorphism would form a k^*-torsor.

When X_1 and X_2 are isomorphic we can fix this flaw, as before, by requiring that $\rho_1^* L_1^{\otimes 2} \simeq \rho_2^* L_2^{\otimes 2}$ over $\pi^* N$. When this is not the case, we have to make precise the statement that $\rho_1^* L_i^{\otimes 2}$'s are isomorphic *in the locus where this makes sense.*

In order to motivate our definition, let us make one final observation. Given $\rho_i : X \to X_i$ and $\rho_i^* L_i$'s on X we can forget about the partial blow-ups $\pi_i : X_i \to C$. Blowing down exceptional components of X on which $\rho_i^* L_i$ has degree 0 would recover $X_i \to C$. This procedure can also be done in families of limit roots.

2.4.2 Multiple limit roots

Assume $m \geq 1$ once again.

Definition 2.17. Let $\pi : \mathcal{X} \to \mathcal{C}$ be a family of blow-ups, $\mathcal{L}$ a line bundle on $\mathcal{X}$ and $\alpha : \mathcal{L}^{\otimes 2} \to \pi^*\mathcal{N}$ a morphism. If there is a limit root $(\pi' : \mathcal{X}' \to \mathcal{C}, \mathcal{L}', \alpha')$ and a factorization $\pi = \pi' \circ \rho$ such that $(\mathcal{L}, \alpha)$ is the pullback of $(\mathcal{L}', \alpha')$ under ρ then we will say that $(\mathcal{L}, \alpha)$ *stabilizes to a limit root* and the map $\rho : \mathcal{X} \to \mathcal{X}'$ is the *partial stabilization with respect to* $\mathcal{L}$.

Remark 2.18. Partial stabilization simply contracts the unstable components of each fiber on which $\mathcal{L}$ has degree 0. See also Lemma I.5.12 to see how this condition behaves.

Definition 2.19. Let $\pi : \mathcal{X} \to \mathcal{C}$ be a family of blow-ups. Let $\mathfrak{L} := \{\mathcal{L}_i, \alpha_i : \mathcal{L}_i^{\otimes 2} \to \pi^*\mathcal{N}\}_{i=1}^m$ be such that each $(\mathcal{L}_i, \alpha_i)$ stabilizes to a limit root, but $\mathfrak{L}$ itself is not pulled back from a partial stabilization. Consider a line bundle $\mathcal{L}$ and a sequence of morphisms $\varphi_i : \mathcal{L} \to \mathcal{L}_i^{\otimes 2}$ satisfying the following:

- $\alpha_i \circ \varphi_i = \alpha_j \circ \varphi_j$ for each i, j.
- Each φ_i restricts to an isomorphism on V_i (see Definition I.5.13).

Then, we will call $\mathfrak{F} = (\varphi_i)_{i=1}^2$ a *synchronization data*. The tuple $(\pi, \mathfrak{L}, \mathfrak{F})$ will be called a *multiple limit root*. An isomorphism of multiple limit roots is an isomorphism of the limit roots commuting with the synchronization data.

Definition 2.20. Let $\overline{\mathcal{S}}^m(\mathcal{N})' \to T$ be the stack associating to $T' \to T$ the groupoid of multiple limit roots of $\mathcal{N}|_{T'}$ over $\mathcal{C}|_{T'} \to T'$.

2.4.3 Relation to torsion-free roots

It remains to show that multiple limit roots (Definition 2.19) and multiple roots (Definition I.5.15) are in fact equivalent. More precisely, we prove the following.

Proposition 2.21. *The two categories $\overline{\mathcal{S}}^m(\mathcal{N})'$ and $\overline{\mathcal{S}}^m(\mathcal{N})$ are equivalent.*

Proof. Given a multiple limit root $(\pi, \mathfrak{L}, \mathfrak{F})$ we can push forward each limit root to obtain a root as in Section 2.3.1. Denote these roots by $\mathfrak{R} = (\mathcal{E}_i, b_i)_{i=1}^m$. Let $D = \bigoplus_{d \geq 0} \pi_* \mathcal{O}_{\mathcal{X}}(d)$ and define $\psi_i : D \to \operatorname{Sym}^{2*} \mathcal{E}_i$ using φ_i. Let $\Psi = (\psi_i)_{i=1}^m$. It is easy to check that $(\mathfrak{R}, \Phi)$ is a multiple root.

Conversely, suppose we are given a multiple root $(\mathfrak{R}, \Phi)$ of $\mathcal{N}$. We will now construct a multiple limit root as follows.

With $\mathfrak{R} = (\mathcal{E}_i, b_i)_{i=1}^m$ consider $\pi_i : \mathcal{X}_i \to \mathcal{C}$ where $\mathcal{X}_i = \underline{\operatorname{Proj}}_{\mathcal{C}}(\operatorname{Sym}^* \mathcal{E}_i)$. Define $\mathcal{L}_i := \mathcal{O}_{\mathcal{X}_i}(1)$ and $\alpha_i : \mathcal{L}_i^{\otimes 2} \to \pi_i^*\mathcal{N}$ as in Section 2.3.1.

The synchronization data $\Psi = (\psi_i : D \to \operatorname{Sym}^{2*} \mathcal{E}_i)_{i=1}^m$ is such that D is a graded sheaf of algebras and the morphisms $\psi_i : D \to \operatorname{Sym}^{2*} \mathcal{E}_i$, which are compatible with b_i's, restrict to isomorphisms on V_i's.

Let $\pi : \mathcal{X} = \underline{\mathrm{Proj}}_{\mathcal{C}} D \to \mathcal{C}$ with $\mathcal{F} := \mathcal{O}_{\mathcal{X}}(1)$. Over V_i where ψ_i is an isomorphism, we get an isomorphism $\rho_{V_i} : \mathcal{X}|_{V_i} \to \mathcal{X}_i|_{V_i}$. Since $\mathcal{X}_i$ is isomorphic to $\mathcal{C}$ away from the discriminant loci contained in V_i, we can extend ρ_{V_i} to $\rho_i : \mathcal{X} \to \mathcal{X}_i$ satisfying $\pi = \pi_i \circ \rho_i$. The morphisms ψ_i yields another morphism $\varphi_i : \mathcal{O}_{\mathcal{X}}(1) \to \rho_i^* \mathcal{O}_{\mathcal{X}_i}(2)$ which restricts to an isomorphism over V_i.

Therefore, if we set $\mathfrak{L} := (\rho_i^* \mathcal{O}_{\mathcal{X}_i}(1), \rho_i^* \alpha_i)_{i=1}^m$ and $\mathfrak{F} = (\varphi_i : \mathcal{O}_{\mathcal{X}}(1) \to \mathcal{O}_{\mathcal{X}_i}(2))_{i=1}^m$ then the tuple $(\pi, \mathfrak{L}, \mathfrak{F})$ is a multiple limit root.

These two constructions are functorial because pushforward and $\underline{\mathrm{Proj}}$ are functorial.

□

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Selbständigkeitserklärung

Ich erkläre, dass ich die Dissertation selbständig und nur unter Verwendung der von mir gemäß § 7 Abs. 3 der Promotionsordnung der Mathematisch-Naturwissenschaftlichen Fakultät, veröffentlicht im Amtlichen Mitteilungsblatt der Humboldt-Universität zu Berlin Nr. 126/2014 am 18.11.2014 angegebenen Hilfsmittel angefertigt habe.

Berlin, den 26. Juni 2017

Emre Can Sertöz